Provision of liquidity to stressed financial markets: Resurrecting the classical lender of last resort

Fabrizio Mattesini
University of Rome Tor Vergata
fabri.mattesini@gmail.com

Ed Nosal
Federal Reserve Bank at Atlanta
ed.nosal@gmail.com

Yu Zhu
Bank of Canada
zhuyuzlf57@gmail.com

This version: February 2023

Abstract

The classical theory of the lender of last resort (Thornton 1802 and Bagehot 1873) emphasizes that the lender: support financial markets and not individuals; behave consistently with longer run (inflation) objectives; and lend freely “to this man and that man” on good collateral at a high rate. Importantly, the classical theory stresses that a lender of last resort is a monetary—and not a credit—operation, where the lender’s objective is to get cash into the hands of people that need it and spend it. These days the classical prescriptions are viewed as outdated and anachronistic—perhaps relevant back in the 1800’s but not in today’s complex, modern financial economy. Instead, the lender of last resort should use its balance sheet to pursue credit and interest rate policies that directly affect long term assets and/or rescue large, interconnected and insolvent institutions, ideas that the classical writers fiercely opposed. We use a standard, dynamic monetary model to assess the classical theory and its prescriptions and find that “lending freely at a high rate on good collateral” enhances social welfare if the economy may be hit by severe liquidity shocks. In fact, some recent policies that central banks, e.g., the Federal Reserve, have pursued in response to aggregate liquidity shortages are consistent with the classical theory.
1 Introduction

When the economy is hit by a severe adverse payments or liquidity shock, the architects of the classical theory of the lender of last resort (Thornton 1802 and Bagehot 1873) emphasized the importance of getting money into the hands of people that need it and spend it so that economic activity does not become severely depressed. And, when the crisis passes, the money should then be quickly taken out of their hands. As a result, they advocated policies that has the lender of last resort lending freely “to this man and that man” on good collateral at a high rate. Among other things, the architects did not view their policy recommendations: (i) as a bailout to failing institutions or individuals;¹ (ii) as some sort of credit or interest rate policy; or (iii) at variance with other objectives of the monetary authority, such as price stability (low inflation). Bagehot (1873) recommended that the policies of the lender of last resort be widely advertised, known to all. Furthermore, it was emphasized that the lender of last resort should not be viewed as a source for everyday, on-going liquidity needs but rather to be tapped on those rare occasions when the markets were significantly stressed or panicked.²

More recently, the classical prescriptions have been viewed in some quarters as outdated and anachronistic (Freixas, Parigi and Rochet 2004), irrelevant (Goodfriend and King 1988) or completely misinterpreted (Humphrey 2010). Frexias, Parigi and Rochet (2004) believe that moral hazard behavior, which is prevalent in banking relationships, undermines the classical theory of the lender of last resort. They and others³ take a more expansive view of the role of a lender of last resort, one that directly engages in credit policy or undertakes risky asset exchanges for the sake of, e.g., saving large, interconnected financial institutions. Mishkin and White (2014) provide a number of examples spanning over 150 years and multiple jurisdictions where central banks have, in fact, behaved in such an expansive manner. Goodfriend and King (1986) claim that central banks’ current focus on targeting short-term, overnight interest rates—e.g., the federal funds rate in the US—eliminates any distinction between monetary policy—i.e., pursuing low inflation—and lender of last resort. And Humphrey (2010), after examining Federal Reserve policy in the aftermath of the Great Financial Crisis, concludes that “[t]he Fed has deviated from the classical model in so many ways as to make a mockery of the notion that it

¹They explicitly recommended against support to insolvent institutions.
²Bignon, Flandreau and Ugolini (2012) provide an excellent overview on Bagehot’s lender of last resort policies from an historical perspective.
is a L[ender] O[f] L[ast] R[esort].”

We agree that some aspects of central banks’ policies resemble fiscal policy and are stark departures from classical lender of last resort policies. But we do not necessarily see that the logical conclusion of moral hazard—which was acknowledged in the times of Thornton and Bagehot—is that central banks need to or should pursue fiscal-type policies (Freixas, Parigi and Rochet 2004) or that classical lender of last resort policies are ineffective or irrelevant. Nor do we think that the distinction between monetary and lender of last resort policies vanish when a central bank policy targets an overnight interest rate. If anything, we think that a monetary policy that targets an overnight rate for every conceivable liquidity shock (Goodfriend and King 1988) is better interpreted as a misguided and non-optimal lender of last resort policy.

Even though the classical policy prescriptions are a response to significant negative economy-wide means of payment shocks, contemporary treatments remain largely silent about this channel. Since, by design, “getting money into the hands of people that need it” is not part of the modern conversation of lender of last resort, it’s perhaps not surprising that recent approaches have, instead, advocated policies that have fiscal policy and/or interest rate control flavors—policies that the classical writers unequivocally disavowed. In this paper we offer a framework that rehabilitates the notion of an economy-wide means of payments shortfall in a way that allows us to assess classical lender of last resort-type policies in a more balanced manner. But in doing so we emphasize that we are not merely taking “a stroll down memory lane” since the mechanisms and policies advocated by classical writers remain both relevant and important today. Hence, they are worthy of discussion. One need not look any further than the chaotic money market events of either September 2019 or March 2022 when financial market participants, including countries, divested themselves of apparently safe, liquid assets—US government treasuries—in favor of a universal means of payments—US dollar reserves. In both cases, the Federal Reserve responded, at least initially, in a manner that appeared generally consistent with the classical prescriptions.

The above discussion re-enforces the idea that any framework intended to study lender of last resort policies should include a medium of exchange that is used for settlement or transactions

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4For example, Humphrey (2010) points out that during and after the 2007-8 financial crisis, the Federal Reserve: (i) focused mainly on credit and not liquidity (money) provision policies; (ii) accepted hard to value collateral; (iii) participated in rescues of insolvent firms (Citigroup and AIG); and (iv) did not withdraw liquidity immediately at the end of the panic.

5I.e., a lender of last resort policy that lends freely but not at a high rate. We have a lot to say about this in the pages that follow.
purposes. Here, we take a modern approach on media of exchange, one that starts with Kiyotaki and Wright (1989), is refined in Lagos and Wright (2005) and extended by many others. Cash investors, those who do not have access to credit, must transact with a tangible means of exchange, fiat money. Credit investors, however, can trade via credit arrangements. At the time when investors make their financial portfolio decisions, they don’t know whether they will have access to credit in the subsequent period. Because of this, all investors end up holding fiat money in their portfolio of financial assets as a “hedge” against not being able to access credit. But notice credit investors have an incentive to lend their money to cash investors, e.g., via repurchase (repo) agreements, since their money balances do not earn any interest income but lending it out might. Cash investors have an incentive to borrow money since that enables them to purchase more investment goods. If the economy is hit by a positive credit “shock,” then most of the investors get credit and market liquidity—measured by the aggregate cash balances of credit investors—will be abundant. In these states, economic activity for the cash investors will be high since the repo borrowing rates will be “low.” In contrast, if the economy is hit by a negative credit shock, market liquidity will be scarce. Economic activity among the cash investors is now low because of the dearth of market liquidity. Repo borrowing rates, which will be “high,” reflect this scarcity. It is in these states that the lender of last resort policies potentially come into play.

Our analysis and quantitative examples give rise to optimal lender of last resort policies that echo the classical prescriptions: lend to anyone with good collateral at high (repo) borrowing rates. The collateral in our environment is riskless government bonds—good collateral—and the lender of last resort’s repo lending rates are significantly higher than the market rates that prevail when market liquidity is not stressed. Bagehot (1893) advocated high borrowing rates to prevent the Bank of England’s gold stock from leaving the country. The rationale for a high repo

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6Hence the title of this paper.
7For a review of this literature, see Rochetau and Nosal (2017). Our model extends ideas developed in Mattesini and Nosal (2016) and Geromichalos and Herrebueck (2016) and shares some features with Berentsen and Waller (2011) and Berentsen and Huber (2014). Duffie et al. (2005, 2007) were the first to take a Kyotaki and Wright (1989) approach to asset markets, which is also relevant for this paper.
8At a superficial level our approach resembles Lucas and Stokey (1987) in the sense that cash is needed to buy some goods while credit suffices for others. Lucas and Stokey (1987) simply impose the restriction that some goods can only be acquired with cash while others can be bought on credit. We, instead, appeal to the existence of monitoring frictions that apply to some investors but not others in determining when money or credit is used.
9Since cash investors face monitoring frictions—they cannot be monitored—and as a result cannot be trusted, any loans they negotiate must be secured by collateral. A repurchase agreement (repo) is a contract where financial securities—the collateral—are effectively sold to the lender by the borrower under the commitment that the borrower purchases them back at some specified future date. It is equivalent to a collateralized loan in which collateral is actually transferred to the lender for the duration of the contract.
rates in our model is similar: in Bagehot, the high borrowing rate ensured that domestic “market liquidity” is not diminished; in our model a high (repo) borrowing rate provides incentives for investors to accumulate real balances and, therefore, higher levels of market liquidity.

The equilibrium outcomes in our environment have a number of interesting and notable features. The optimal policy has investors borrowing from the lender of last resort only occasionally. In fact, the vast majority of the time, the volume of market liquidity is sufficient to carry out transactions. The lender of last resort could, of course, lower its lending rate which could increase economic activity in stressed states. However, such a move would exacerbate the moral hazard problem that the classical writers seek to contain. In particular, a lower borrowing rate would reduce the incentives for investors to accumulate the medium of exchange and, as a result, would also reduce market liquidity. In this situation, investors’ reliance on the lender of last resort would increase. And in states of the world where market liquidity is not stressed, economic activity falls—compared to one with higher borrowing rates—because market liquidity is lower. When choosing its lending rate, the lender of last resort faces a tradeoff: lowering the lending rate increases activity in the liquidity stressed states but at the expense of lower economic activity in non-stressed states. Our two-state example highlights the reluctance of the lender of last resort to open the door to moral hazard: unless market liquidity is very, very scarce, the lender of last resort has no incentive to augment a liquidity shortage by lending. And when it does choose to provide liquidity support in this very (very) scarce market liquidity state, it does so at a very high borrowing rate.

Bagehot (1873) advocated that lender of last resort policies should be well advertised. We find that society will be better off when the lender of last resort’s policy is explicit and well advertised—e.g., the central bank always lends at $x\%$ on good collateral—as opposed to uncertain—e.g., if overnight market rates remain highly elevated for some period of time, then it will lend at $y\%$ on good collateral. The classical writers also claimed that there should be no conflict between a central bank’s role as lender of last resort and its other mandates, such as pursuing price stability. There is a view that lender of last resort policies have an inflationary bias which means these policies are at variance with a central bank’s monetary policy mandate, such as price stability. Our analysis finds no such conflict. Since a lender of last resort policy increases the economy-wide medium of exchange when market liquidity is severely stressed and then removes it when the crisis passes, lender of last resort policies are independent of and can be completely separated from a central bank’s price stability mandate.
Our paper focuses on a particular type of moral hazard, which is related to market liquidity and the lender of last resort’s lending rate. More specifically, when the lending rate is lowered, investors accumulate less media of exchange and, as a result, market liquidity declines with the lender of last resort’s lending rate. The more recent (industrial organization-motivated) literature on the lender of last resort—which suppresses/ignores the classical theory’s focus on the medium of exchange—instead examines “liquidity” issues that arise from intermediaries’ investment decisions when investment choices are subject to asymmetric information. Here, an intermediary may need additional output in an early period to continue as a going concern.\footnote{Just as in Diamond and Dybvig’s (1983) finite period, 3 date model environment, where investments are undertaken in period 0, can be partially liquidated in date 1 and fully mature in period 2, this literature defines liquidity as date 1 output.} The moral hazard problem, i.e., the choice of the intermediary’s investment and/or monitoring of it, combined with asymmetric information implies that a potential lender does not know the quality of the intermediary’s collateral, where the collateral is the investment project. That is, the lender of last resort may not know if the intermediary needs to borrow because it experienced a liquidity event and has good collateral or because it is insolvent, i.e., has bad collateral, and wants to borrow in hopes that it might survive. If the lender of last resort extends a loan, the collateral that it takes will be “unsound” or risky, i.e., there is a chance that the collateral’s value is less than the loan. A few comments are in order. First, it is a somewhat strained interpretation to think of the lender of last resort as a central bank in this literature. In reality, a central bank lends (or provides repo finance) by swapping fiat money or reserves for an electronic security—such as US government treasuries. As a result, the amount that a central bank can lend is constrained only by the amount of available collateral. In this literature, the central bank needs real resources, i.e., a warehouse of date 1 goods, in order to play the role of lender of last resort and, therefore, will be constrained by its own resources. Second, when the central bank accepts risky, and possibly insolvent, collateral, it is engaging in a type of fiscal policy.\footnote{Interestingly, Bignon, Flandreau and Ugolini (2012) find that the quality of collateral increased when the Bank of England began pursuing lender of last resort policies.} In our (classical) framework the central bank only lends on good collateral and lending simply increases the amount of liquidity available to the economy used to purchase goods. That being said, our model could be extended to allow for risky investments that are subject to the kind of moral hazard and asymmetric information described above. The investor now holds a portfolio of money, government bonds and risky real investment. The central bank is still able to provide liquidity on good collateral, i.e. government bonds, to the investor. The new issue
that arises here is whether a central bank should provide a loan on good collateral if it suspects that the investor’s overall portfolio is insolvent. This interesting issue is outside of the scope of current paper but it illustrates how the more recent industrial organizational treatments of lender of last resort can be addressed within the context of a model where the lender of last resort provides the medium of exchange as part of the lending arrangement.

The remainder of the paper is organized as follows. Section 2 lays out the benchmark model where the aggregate liquidity shock takes on only two values. This setup allows us to convey the intuition of the main results in a simple and transparent way. Section 3 examines equilibrium outcomes when the lender of last resort role is always inactive. This section illustrates how private markets deal with aggregate liquidity shocks. Section 4 introduces a (sometimes) active lender of last resort and shows how it can improve welfare. Section 5 extends the model by allowing a general distribution of the aggregate liquidity shock and shows that our insights and results are robust to this generalization. Section 6 considers conditions under which a standing repo facility—one that is always available—is better than or worse than an emergency repo facility—one is brought “on line” after significant market illiquidity is observed. Section 7 concludes. All proofs and other technical details can be found in the appendix.

2 Benchmark model

Cash investors need money to purchase investment goods. They can get more money—to buy more investment goods—by either selling their assets or using them as collateral for repo finance. Credit investors can finance investments using credit. As a result, they are able to contribute to market liquidity by using their idle balances to either purchase assets outright or provide repo finance. Market liquidity will vary with credit conditions, where credit conditions are measured by the fraction of credit investors in the economy. When credit conditions and, hence, market liquidity deteriorate, a central bank can support the economy’s liquidity needs by providing repo finance to cash investors. The central bank’s repo facility essentially “buys” assets from cash investors with newly issued cash with the promise that the investors repurchase them in a near future date. We now turn to the details of the model.

Investors hold a portfolio of one-period government bonds, \( b \), and real money balances, \( z \). A government bond pays one unit of a real consumption good at maturity and money is a fiat, 

\[12\] A repo lender provides cash to a repo borrower and secures the loan with the borrower’s collateral. The repo borrower promises to pay back the cash, plus an additional amount that represents interest, in return for the pledged collateral at a future, prespecified date.
nominal object. The government sells \( \bar{b} \) one-period bonds each period. Bond repayments are financed by a lump-sum tax \( T_b \), where \( T_b = \bar{b} \) since we normalize the measure of investors to 1. A fraction \( \sigma \) of investors are randomly chosen to be cash investors and the remainder are credit investors. For expositional simplicity, we initially assume the credit shock \( \sigma \) takes on 2 values, \( \sigma_L \) and \( \sigma_H \), where \( 0 < \sigma_L < \sigma_H < 1 \). We later extend the analysis to allow for a more general distribution. State \( H \) corresponds to a deterioration of credit conditions since \( \sigma_H > \sigma_L \). State \( L \) (\( H \)) occurs with probability \( \pi_L \) (\( \pi_H \)), where \( 0 \leq \pi_L, \pi_H < 1 \) and no aggregate risk when \( \pi_L, \pi_H = 0, 1 \).

Investors are infinitely lived and time is indexed by \( t \). Each time period \( t \) has 3 subperiods. The first subperiod is the financial market or **FM subperiod**. Investors enter the FM subperiod with portfolio \((b, z)\) and learn whether they are cash or credit investors. Investors can adjust their liquidity and asset positions in a competitive financial market and with the central bank. Cash investors can either sell bonds or get repo finance—using their bonds as collateral—in the financial market. We assume that investors incur a very small transactions cost when selling or buying assets in the FM subperiod, which implies that investors will use repo arrangements to obtain additional real balances or bonds. A repo contract in state \( i = L, H \) specifies two prices, \( p_{iFM} \) and \( p_{iR} \), where \( p_{iFM} \) is the competitive repo price per unit of government bond (or collateral) measured in terms of real balances and \( p_{iR} \geq p_{iFM} \) is the price at which cash investors repurchase their collateral in a subsequent (third) period. Hence, a cash investor can exchange collateral \( b^c_i \leq b \) for \( p_{iFM}^c b^c_i \) real balances. A credit investor provides \( z^n_i \leq z \) real balances in repo finance which is secured by \( z^n_i / p_{iFM}^c \equiv b^n_i \) collateral, where the superscript \( n \) denotes that the investor is not a cash investor. Demand for market liquidity—or, equivalently, aggregate demand for real balances—is \( \sigma_i p_{iFM}^c b^c_i \) and supply of market liquidity is \((1 - \sigma_i) z^n_i \). The central bank provides liquidity in the FM subperiod through a repo facility. At the beginning of each period, before the state is revealed, the central bank repo facility posts two non-state contingent prices, \( p^{CB} \) and \( p^R \), and stands ready to purchase government bonds in any amount.

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\(^{13}\) For simplicity and without loss of generality, we assume that the non-monetary asset is a one-period real government bond. The qualitative nature of our results are unaffected if we assume that the asset is a nominal government bond or a Lucas tree.

\(^{14}\) We examine a more general specification in Section 5 and Appendix D, where \( \sigma \) is identically and independently distributed over \([0, 1]\).

\(^{15}\) Similar to Duffie (1996), we model the repo market as a competitive one.

\(^{16}\) More generally, we assume that the transactions cost associated with repo finance is less than that associated with selling and buying assets, a condition that holds in practice. If the transactions cost associated with repo finance and selling and buying assets are equal, then investors would be indifferent between repo transactions and buying and selling assets.
from investors at repo price $p^{CB}$. The repo arrangement requires that investors repurchase their bonds in a subsequent (third) subperiod at the repurchase price $p^R \geq p^{CB}$. The FM subperiod competitive market repo rate in state $i = L, H$ is defined as $r^{FM}_i \equiv (p^R_i - p^{FM}_i)/p^{FM}_i$ and the central bank repo rate is $r^{CB} \equiv (p^R - p^{CB})/p^{CB}$. If a cash investor repo finances collateral equal to $b^c_i$ in the competitive financial market and $b^{CB}_i$ at the central bank’s repo facility, then his real balance holdings increase by $p^{FM}_ib^c_i + p^{CB}b^{CB}_i$ and his (implicit) bond holdings decrease by $b^c_i + b^{CB}_i$. Feasibility requires $b^c_i + b^{CB}_i \leq b$. Cash investors exit the FM subperiod holding portfolio $(b - b^c_i - b^{CB}_i, z + p^{FM}_ib^c_i + p^{CB}b^{CB}_i)$ while credit investors hold $(b + p^Riz^i/p^{FM}_i, z - z^R)$.\footnote{It is convenient to represent a repo transaction as a change in real money balances in the FM subperiod along with the change in the bond holding adjusting for the repurchase price, $p^R$ or $p^{CB}$. For example, in the case of the cash investor, his real balances increase by $p^{FM}_ib^c_i + p^{CB}b^{CB}_i$ in the FM subperiod, but in order to get this increase in real balances, he must repurchase in total $b^c_i + b^{CB}_i$ bonds in the subsequent CM subperiod from the repo counterparties at a total value greater than the cash he initially received.}

In the second subperiod, cash and credit investors bargain with sellers over the amount, $y^j_i$, and total price, $p^j_i$, of investment goods to exchange in state $i = L, H$, where $p^j_i$ is measured in real balances and $j = c, n$. In this decentralized investment market or DM subperiod, the cash investor pays the seller $p^c_i$ real balances in this subperiod while the credit investor pays the seller $p^n_i$ consumption goods in the subsequent subperiod.\footnote{Real balances are measured in terms of the third subperiod consumption good.} The latter is a credit arrangement since the seller extends a loan, $y_i$, to be repaid, $p^R$, in a later subperiod. Sellers are infinitely lived and their measure is at least equal to 1.\footnote{Without loss of generality, we assume that sellers do not participate in the FM subperiod financial markets. We elaborate on this below.} Sellers, and only sellers, can produce perishable investment goods using their labor in a linear technology, where a unit of labor produces a unit

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure1.png}
\caption{Timing of events}
\end{figure}
of the investment good. The seller’s disutility of y labor is \( c(y) \), where \( c(0) = 0 \), \( c' > 0 \) and \( c'' \geq 0 \). The investment good is used in technology \( f \) that is only available to investors. The technology generates \( f(y) \) units of a consumption good in the next subperiod, where \( f(0) = 0 \), \( f' > 0 \), \( f'' < 0 \) and \( f'(0) > c'(0) \). The market structure in the DM subperiod has each investor being matched with a seller. In each match, investors and sellers bargain over the terms of trade \( (p^j_i, y^j_i) \), \( j = c, n \). We assume that \( (p^j_i, y^j_i) \) is determined by the Kalai bargaining solution.

Intuitively, if the investor has bargaining power \( \theta \) and \( S \) represents total surplus generated by the investor-seller match, \( p^j_i \) and \( y^j_i \) are set so that the investor and seller receive \( \theta S \) and \( (1 - \theta)S \), respectively. Feasibility for the cash investor requires that \( p^c_i \leq z + p^{FM}_i b^c_i + p^{CB} b^{CB}_i \), where the right side is the total real balances held by the cash investor. Cash investors exit the DM subperiod holding the net portfolio \((b - b^c_i - b^{CB}_i, z + p^{FM}_i b^c_i + p^{CB} b^{CB}_i - p^c_i)\) while credit investors exit with the same portfolio of money and bonds they entered the DM with, \((b + p^R_i z^n_i / p^{FM}_i, z - z^n_i)\) along with the credit obligation \( p^n_i \). Matched sellers exit holding either \( p^c_i \) real balances or \( p^n_i \) worth of real credit.

In the third and final subperiod, which we call the competitive market or \textit{CM subperiod}, there exists an Arrow-Debreu market where investors and sellers trade money for the consumption good, investors pay off their debts—to sellers, the central bank and other investors—and rebalance their portfolio of assets they intend to take into the next period. The government levies the lump-sum tax, \( T_b \), on investors and pays off its one-period debt obligations. Each investor receives a payoff from his investment equal to \( f(y^j_i) \) units of the consumption good, \( j = c, n \), at the very beginning of the CM subperiod. Investors that entered into a repo contract with the central bank and/or other investors in the previous FM repurchase their collateral at the stated price—\( p^R \) for central bank repo and \( p^R_i \) for competitive repo—with real balances. If \( \phi \) represents the amount of consumption goods that 1 unit of fiat money can buy in the CM subperiod competitive market, then \( 1/\phi \) is the price of the consumption good measured in terms of money, e.g., dollars. Hence, if an agent holds \( m \) units of nominal balances, then real balance holdings are simply \( z = \phi m \). The price of the newly issued one-period government bonds in the CM subperiod is denoted by \( p^{CM} \), measured in terms of the consumption good (or real balances). Sellers and investors have linear, one-to-one preferences over the consumption good. The representative investor exits the CM subperiod of period \( t \) and enters period \( t + 1 \) with portfolio \((b_+, z_+)\).\(^{20}\) See Figure 1 for a summary of the timing of events in a typical period \( t \).

\(^{20}\)Owing to the specification of preferences, the CM subperiod asset price does not depend on the state \( i \) and
In practice, repo finance is designed to provide the lender with some insurance against default and price change. This is typically accomplished by having the repo lender provide a cash loan that is less than the market value of the collateral, and is referred to as applying a “haircut” to the collateral. In the event of a default, the haircut helps the repo lender recover the total loan repayment, principle and interest, in most circumstances. Because there is no uncertainty regarding the payoff of the one period government bond in the CM—each bond pays one unit—and the repo lender possesses the asset, any haircut will yield the same equilibrium. Therefore, without loss of generality we assume there is no haircut and that \( p^R_R = 1 \).

Since the repurchase price is essentially “predetermined,” the competitive and central bank repo contracts are fully described by the FM subperiod repo prices, \( p^{FM}_i \) and \( p^{CB}_i \), respectively. Notice that the central bank repo facility will be inactive in state \( i \) whenever \( p^{CB}_i < p^{FM}_i \) since cash investors strictly prefer getting repo finance in the competitive financial market.

In addition to operating a repo facility, the central bank sets an inflation target, \( \pi^* \). Define \( M_t \) to be the aggregate stock of nominal money at the beginning of period \( t \). Then \( M_{t+1} = \mu_t M_t \), where \( \mu_t \) represents the gross growth rate in aggregate nominal money balances between periods \( t \) and \( t+1 \). Since, in practice, central banks are reluctant to pursue a deflationary policy and do not have taxing authority, we have \( \mu_t \geq 1 \). New money is injected into the economy by lump-sum transfers \( T^M_i \) to investors at the beginning of the CM subperiod of period \( t \). If \( \phi_t \) is the price of money in period \( t \), then the value of aggregate real money balances in the CM subperiod of period \( t \) is \( \phi_t (M_t + T^M_i) \).

In the language of Holmstrom and Tirole (1998), credit investors are able to *pledge* all of investment income, \( f(y_i) \), while cash investors cannot pledge any of it. Whether investors are able to pledge their investment income depends on being monitored. Credit investors can pledge all of their investment income because they are monitored in the DM and CM subperiods of period \( t \). If a credit investor attempts to default, he is known and can be “found” and his output can be confiscated to settle the debt. Cash investors, on the other hand, cannot be monitored in either the DM or CM subperiods of period \( t \). Sellers will not extend credit to cash investors since they are unknown and cannot be found. If credit was extended, cash investors would costlessly exit the CM subperiod of period \( t \) holding the same portfolio, independent of whether the investor was a cash or credit investor in period \( t \). Since sellers do not participate in the FM subperiod financial markets, their demands for real balances and real assets in the CM subperiod are zero.

\(^{21}\) The repo lender will not recover the total loan repayment if, for example, the market value of the collateral experiences a significant decline.

\(^{22}\) That credit investors can pledge all of their investment income implies that \( p^i_i \leq f(y_i) \), which is always the case in equilibrium.
default on their obligations. Hence, none of a cash investor’s investment income is pledgeable.

Notice that credit investors and the central bank will “lend” to cash investors via a repo arrangement since the repo loan is secured—or “tangibly pledged”—by government bonds whose value is at least equal to the value of the repo loan. We assume that ownership of the government bond is digitally stored at a repository that can only be accessed when financial markets operate, which is in the FM and CM subperiods. Since government bond ownership cannot be verified and transferred in the DM subperiod, bonds cannot serve as a medium of exchange in the DM subperiod. This implies that any transfer of investment goods from sellers to cash investors must be settled in cash in the DM subperiod.

3 An always inactive central bank repo facility

Here we assume that the competitive financial market is the sole source of liquidity in the FM subperiod. This occurs, for example, if \( p^{CB} < \min\{p^{FM H}, p^{FM L}\} \), which implies that, in equilibrium, the central bank repo facility is always inactive.\(^{23}\) We parameterize \( \sigma_L \) and \( \sigma_H \) in a way that highlights the intuition that underlies the potential costs and benefits associated with an active central bank repo facility.\(^{24}\) Our parameterization gives rise to an equilibrium that is characterized by:

- in state \( i = L \), \( \sigma_L \) cash investors repo finance all their collateral (i.e., government bond holdings) in the FM subperiod competitive financial market, \( b^C_L = b \), and \( 1 - \sigma_L \) credit investors do not supply all of their real balances for repo finance, \( z^n_L < z \); and
- in state \( i = H \), \( \sigma_H \) cash investors repo finance only a fraction of their collateral in the FM subperiod competitive financial market, \( b^C_H < b \), and \( 1 - \sigma_H \) credit investors supply all of their real balances for repo finance, \( z^n_H = z \).

Intuitively, when \( \sigma = \sigma_L \) credit conditions are “loose” and market liquidity is abundant in the sense that: (i) there is cash equal to \( (1-\sigma_L)(z-z^n_L) \) sitting on the “sidelines,” i.e., staying in the portfolios of credit investors and (ii) cash investors are able to repo finance all of their collateral. When \( \sigma = \sigma_H \) credit conditions are “tight” and market liquidity is scarce in the sense that: (i)\(^{23}\) Setting \( p^{CB} = 0 \) ensures that the central bank’s repo facility is always inactive.

\(^{24}\)Our parameterization is in no way “contrived.” A more general model, which is presented and analyzed in Section 5 and Appendix D, has three kinds of equilibria that emerge in the FM subperiod, two of which are described by our parameterizations of \( \sigma_L \) and \( \sigma_H \). The third equilibrium configuration is basically a mixture of the two that are presented in this section.
there is no cash sitting on the sidelines and (ii) cash investors hold some government bonds in their portfolio. Such an equilibrium configuration arises when $\sigma_L$ is sufficiently small and $\sigma_H$ is sufficiently large. We now provide details about investors’ and sellers’ decision making so that we can fully characterize the equilibrium.

The value function for an investor at the beginning of the CM subperiod, $W(b, z, y, d)$, is given by

$$W(b, z, y, d) = \max_{x, b', z'} \{ x + \beta \mathbb{E}_i J(b', z', \sigma_i) \},$$

s.t. $x + p^C M b' + \phi / \phi' z' + d = b + z + \phi T^M + f(y) - T_b$

where $b$ and $z$ are the amounts of government bonds and real balances—both net of repo transactions—brought into the CM subperiod, $y$ is the amount of the investment good invested in DM subperiod, $d$ is the real credit obligation to a seller (which equals $p^n$ for a credit investor and zero for a cash investor), $x$ is the amount consumed of the real consumption good, $b'$ and $z'$ are government bonds and real balances, respectively, brought into the next period, $\phi'$ is the price of one unit of fiat money in the next period, $T^M$ is the central bank’s lump sum nominal monetary transfer, $T_b$ is the government’s real lump-sum tax and $J(b', z', \sigma_i)$ is the investor’s value function at the beginning of the subsequent FM subperiod when the state of the world is $i = H, L$. The expectation is taken with respect to the state in the next period, either $i = H$ or $L$. We can eliminate $x$ from the CM subperiod value function using the budget constraint to get

$$W(b, z, y, d) = f(y) + b + z + \phi T^M - d - T_b + \max_{b', z'} [-p^C M b' - \phi / \phi' z' + \beta \mathbb{E}_i J(b', z', \sigma_i)].$$

The first-order (Euler) conditions, assuming interior solutions, are

$$b' : p^C M = \beta \mathbb{E}_i J_1 (b', z', \sigma_i), \quad (1)$$

$$z' : \frac{\phi}{\phi'} = \beta \mathbb{E}_i J_2 (b', z', \sigma_i), \quad (2)$$

and the envelope conditions are

$$W_1 (b, z, y, d) = W_2 (b, z, y, d) = 1, W_3 (b, z, y, d) = f'(y) \text{ and } W_4 (b, z, y, d) = -1.$$

Intuitively, owing to the linearity in preferences over the consumption good, investors value both an additional unit of the government bond that matures in that CM subperiod and an
additional unit of real balances at one, the amount of the consumption good that they can
purchase. Linearity also implies

\[ W(b, z, y, d) = f(y) - d + W(b, z, 0, 0). \]  

The value function at the beginning of the FM subperiod in state \( i = L, H \), \( J(b, z, \sigma_i) \), is
given by

\[ J(b, z, \sigma_i) = \sigma_i J^c(b, z, \sigma_i) + (1 - \sigma_i) J^n(b, z, \sigma_i), \]

where \( J^c(b, z, \sigma_i) \) is the cash investor’s state \( i \) value function, \( J^n(b, z, \sigma_i) \) is the credit investor’s
state \( i \) value function and \( b \) and \( z \) represent the bond and real balance holdings at the beginning
of the FM subperiod.

In the FM subperiod cash investors use their collateral to obtain additional real balances so
they can purchase more investment goods in the DM subperiod. Since the financial market in
the FM subperiod is competitive, cash investors take the bond repo price, \( p_{FM}^c \), as given and solve

\[ J^c(b, z, \sigma_i) = \max_{b_c \leq b} V^c(b - b_c, z + z_c, \sigma_i) \]

s.t. \( p_{FM}^c b_c = z_c \),

where \( V^c \) is cash investor’s value function in the DM subperiod and \( b_c \leq b \) is the repo collateral
constraint, i.e., a cash investor can’t repo finance collateral he does not hold.

We assume that investors get matched with sellers with probability 1 in the DM subperiod.
The terms of trade in a match are determined by Kalai bargaining, where the investor has
bargaining power \( \theta \).\(^{25} \) If a cash investor enters the DM subperiod with real balances \( z_i \) in
state \( i \), the quantity of investment good produced \( y_i^c \) and real payment \( p_i^c \) for those goods are
determined by,

\[ \max_{y_i^c, p_i^c \leq z_i} \left[ f(y_i^c) - p_i^c \right] \]

s.t. \( f(y_i^c) - p_i^c = \theta \left[ f(y_i^c) - c(y_i^c) \right] \),

i.e., the terms of trade maximize the cash investor’s surplus subject to the investor getting a
fraction \( \theta \) of the total match surplus and a cash constraint. The amount of investment good
produced and payment for it expressed as functions of \( z_i \), \( Y(z_i) \) and \( P(z_i) \), respectively, are

\[
Y(z_i) = \begin{cases} v^{-1}(z_i) & \text{if } z_i = p_i^c < v(y^*) \\ y^* & \text{otherwise} \end{cases},
\]

\[
P(z_i) = \begin{cases} z_i & \text{if } z_i = p_i^c < v(y^*) \\ z^* = v(y^*) & \text{otherwise} \end{cases}.
\]

\(^{25}\) The same analysis can be applied to a general trading mechanism as in Gu and Wright (2016).
where \( v(y) = (1 - \theta) f(y) + \theta c(y) \), \( y^* \) is the efficient level of the investment good, i.e., \( f'(y^*) = c'(y^*) \).\(^{26}\) We assume that \( b \), the amount of government bonds that investors hold at the beginning of the FM subperiod, is not “large” in the sense that even if the cash investor repo finances all of his collateral in the FM subperiod market in state \( i = L \), total real balance holdings are strictly less than \( v(y^*) \), the amount of real balances needed to purchase the first-best level of the investment good, \( y^* \).\(^{27}\) One can express the cash investor’s DM subperiod value function in state \( i \) as

\[
V^c(b_i, z_i, \sigma_i) = W[b_i, z_i - P(z_i), Y(z_i), 0].
\]  

(5)

Intuitively, the matched investor buys \( Y(z_i) \) investment goods and pays \( P(z_i) \) for them if he has a total of \( z_i \) real balances at the beginning of the DM subperiod.

Combining (4) and (5), and using (3) we get

\[
J^c(b, z, \sigma_i) = \max_{b_\ell \leq b} \{ f[Y(z + z_i^c)] + W(b - b_\ell, z + z_i^c - P(z + z_i^c), 0, 0) \}
\]

s.t. \( p_i^{FM} b_\ell^c = z_i^c \),

(6)

Exploiting the linearity of \( W \) and using the budget constraint to eliminate \( z_i^c \), (6) can be rewritten as,

\[
J^c(b, z, \sigma_i) = \max_{b_\ell \leq b} \{ f[Y(z + p_i^{FM} b_\ell^c)] - P(z + p_i^{FM} b_\ell^c) - b_\ell^c (1 - p_i^{FM}) + W(b, z, 0, 0) \}.
\]

(7)

The first-order condition for the right-side of (7) is

\[
\lambda(z + p_i^{FM} b_\ell^c) \begin{cases} 
(1 - p_i^{FM}) / p_i^{FM} & \text{if } b_\ell^c < b \\
\geq (1 - p_i^{FM}) / p_i^{FM} & \text{if } b_\ell^c = b
\end{cases},
\]

(8)

where

\[
\lambda(z + p_i^{FM} b_\ell^c) \equiv \frac{f'[Y^c(z + p_i^{FM} b_\ell^c)]}{v'[Y^c(z + p_i^{FM} b_\ell^c)]} - 1 = \frac{\theta \{ f' \left[ Y^c \left( z + p_i^{FM} b_\ell^c \right) \right] - c' \left[ Y^c \left( z + p_i^{FM} b_\ell^c \right) \right] \}}{v' \left[ Y^c \left( z + p_i^{FM} b_\ell^c \right) \right]}. \quad (9)
\]

\( \lambda(\cdot) \) can be interpreted as liquidity premium for real balances in the FM subperiod and \((1 - p_i^{FM}) / p_i^{FM}\) represents the marginal cost of (converting collateral into) real balances.

---

\(^{26}\) Notice that if the cash constraint does not bind, then the solution to the above maximization problem is \( f'(y^*) = c'(y^*) \) and the cash investor pays \( v(y^*) = (1 - \theta) f(y^*) + \theta c(y^*) \) real balances for \( y^* \) units of the investment good.

\(^{27}\) In Appendix E we examine the case where \( b \) is “large” in the sense that \( z + p_i^{FM} b_\ell^c = z^* \) in the liquidity abundant state \( i = L \), where \( b_\ell^c \leq b \). That is, in state \( i = L \) the buyer is able to purchase the efficient amount of the investment good \( y^* \). When \( b \) is “large,” market liquidity will be scarce in state \( H \) when \( \sigma_H \) is sufficiently large.
A credit investor’s state \( i \) value function in the FM subperiod is given by

\[
J^n(b, z, \sigma_i) = \max_{z_i^n \leq z} W(b + b_i^n, z - z_i^n, 0, 0) + f(y^*) - v(y^*),
\]

s.t. \( p_{i}^{FM}b_i^n = z_i^n \),

Since credit investors are not constrained by a means of payment in the DM subperiod—their investment output is fully pledgeable—they negotiate an outcome with a seller that maximizes total surplus \( f(y) - c(y) \), which implies that \( p_i^n \equiv v(y^*) \) and \( y = y^* \). Credit investors can use their real balances to provide repo finance in the FM subperiod competitive market. Again, exploiting the linearity of \( W \) and using the budget constraint to eliminate \( b_i^n \), \( J^n(b, z, \sigma_i) \) can be written as

\[
J^n(b, z, \sigma_i) = \max_{z_i^n \leq z} \left( \frac{1}{p_{i}^{FM}} - 1 \right) z_i^n + W(b, z, 0, 0) + f(y^*) - v(y^*)
\]  

(10)

The solution to the right-side of (10) is

\[
z_i^n \begin{cases} 
\in (0, z) & \text{if } p_{i}^{FM} = 1 \\
= z & \text{if } p_{i}^{FM} < 1.
\end{cases}
\]

Intuitively, a credit investor is indifferent between providing and not providing repo finance in the FM subperiod when the FM subperiod repo bond price, \( p_{i}^{FM} \), equals 1, the payoff of a government bond in the subsequent CM subperiod. In other words, the competitive repo rate is zero. However, when \( p_{i}^{FM} \) is less than one, a credit investor supplies all of his real balances for repo finance since the excess payoff per unit of collateral supplied, \( 1 - p_{i}^{FM} \), is strictly positive. Notice that \( p_{i}^{FM} < 1 \) necessarily implies that \( \lambda(\cdot) > 0 \), see (8). An immediate implication is that, in equilibrium, we must have \( p_{i}^{FM} \leq 1 \). If \( p_{i}^{FM} > 1 \), then the supply of repo finance will be zero and the demand will be strictly positive; hence, the FM subperiod financial market will not clear.

We can now use (1) and (2) to determine the equilibrium CM subperiod price of the newly issued government bonds and demand for real balances. Since there are only two states, \( L \) and \( H \), it will be convenient to express (1) and (2) as

\[
p^{CM} = \pi_L \beta[\sigma_L J_1^L (b, z, \sigma_L) + (1 - \sigma_L) J_1^n (b, z, \sigma_L)]
\]

\[
+ \pi_H \beta[\sigma_H J_1^H (b, z, \sigma_H) + (1 - \sigma_H) J_1^n (b, z, \sigma_H)]
\]  

(11)

and

\[
\frac{\phi}{\phi'} = \pi_L \beta[\sigma_L J_2^L (b, z, \sigma_L) + (1 - \sigma_L) J_2^n (b, \sigma_L)]
\]

\[
+ \pi_H \beta[\sigma_H J_2^H (b, z, \sigma_H) + (1 - \sigma_H) J_2^n (b, z, \sigma_H)],
\]  

(12)
respectively. When $i = L$, credit conditions are loose (or equivalently market liquidity is abundant) and cash investors repo finance all of their collateral in the FM subperiod. The former implies that credit investors are indifferent between holding cash and providing repo finance so $p^F_{LM} = 1$; the latter implies that $b^c_L = b$. Hence, the cash investor’s FM subperiod value function (7) in state $L$ is

$$J^c(b, z, \sigma_L) = f \circ Y(z + b) - P(z + b) + W(b, z, 0, 0),$$

and the envelope conditions imply

$$J^c_1(b, z, \sigma_L) = \lambda(z + b) + 1,$$
$$J^c_2(b, z, \sigma_L) = \lambda(z + b) + 1. \quad (13)$$

The credit investor’s FM subperiod value function (10) becomes

$$J^n(b, z, \sigma_L) = W(b, z, 0, 0) + f(y^*) - v(y^*),$$

and we have

$$J^n_1(b, z, \sigma_L) = 1,$$
$$J^n_2(b, z, \sigma_L) = 1. \quad (15)$$

When $i = H$, market credit conditions are tight (or equivalently market liquidity is scarce). Since cash investors do not repo finance all of their collateral in the FM subperiod financial market, let $b^c_H$ be the amount of collateral that a cash investor repo finances, where $b^c_H$ solves (8) with equality. Hence, the cash investor’s FM subperiod value function (7) in state $H$ is

$$J^c(b, z, \sigma_H) = f \circ Y(z + p^F_{HM}b^c_H) - P(z + p^F_{HM}b^c_H) - b^c_H(1 - p^F_{HM}) + W(b, z, 0, 0),$$

and we have

$$J^c_1(b, z, \sigma_H) = 1,$$
$$J^c_2(b, z, \sigma_H) = \lambda(z + p^F_{HM}b^c_H) + 1. \quad (17)$$

Since $z^n_H = z$, the credit investor’s value function (10) when $i = H$ can be written as

$$J^n(b, z, \sigma_H) = \left(\frac{1}{p^F_H} - 1\right) z + W(b, z, 0, 0) + f(y^*) - v(y^*),$$

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and we have

\[ J_1^n (b, z, \sigma_H) = 1, \tag{19} \]
\[ J_2^n (b, z, \sigma_H) = \frac{1}{p_{FM}^H} = \lambda(z + p_{FM}^H b^*_H) + 1, \tag{20} \]

where the second equality in (20) follows from (8) with equality.

We consider a competitive steady-state equilibrium with rational expectations. The steady state equilibrium requirement means that real variables—such as \( z_t \), \( p_{CM}^t \), \( \phi_t(M_t + T_t^M) \), and so on—are unchanging over time. The competitive equilibrium requirement means that supply equals demand and rational expectations means that agents’ forecasts are consistent with equilibrium outcomes. Substituting (13)-(20) into (11)-(12), imposing the steady state conditions and market clearing for government bonds in the CM subperiod, \( b = \bar{b} \), we get\(^{28}\)

\[ p_{CM}^t = \beta + \beta \pi_L \sigma_L \lambda(z + \bar{b}), \tag{21} \]

and

\[ \frac{\phi}{\phi'} - 1 = \pi_L \sigma_L \lambda(z + \bar{b}) + \pi_H \lambda(z + p_{FM}^H b^*_H). \tag{22} \]

Since equilibrium in the FM subperiod in state \( i = H \) requires \( \sigma_H b^*_H p_{FM}^H = (1 - \sigma_H) z \) or \( z + p_{FM}^H b^*_H = z/\sigma_H \), we can rewrite (22) as

\[ \frac{\phi}{\phi'} - 1 = \pi_L \sigma_L \lambda(z + \bar{b}) + \pi_H \lambda(z/\sigma_H). \tag{23} \]

Three observations are in order. First, the so-called Fisher equation relates the nominal interest rate (on an illiquid one period bond), denoted as \( \iota \), to the real interest rate \( r \) and inflation rate \( \phi/\phi' - 1 \). More specifically, the Fisher equation is

\[ \iota = \frac{\phi}{\phi'} - 1. \tag{24} \]

Hence, the left sides of (22) and (23) can be interpreted as a nominal interest rate. Second, since the fundamental value for a newly issued government bond in the CM subperiod is \( \beta \) and for real balances is zero,\(^{29}\) (21) and (23) indicate that asset prices exceed their fundamental values, i.e., assets have a liquidity premium in the CM subperiod. Third, the equilibrium can be solved sequentially. The right side of (23) is decreasing in \( z \) and becomes negative as \( z \) gets arbitrarily

\(^{28}\)The market for real balances clear if \( z'/\phi' = M_t + T_t \), which we fully characterize below.

\(^{29}\)The discounted value of real bond payments is sometimes called the the bond’s fundamental value. A one-period government bond that pays one unit of the consumption good has a fundamental value of \( 1/(1 + r) \equiv \beta \). Since fiat money does not provide any interest or dividends, its discounted stream or fundamental value is zero.
large and approaches \(\theta/(1 - \theta)\) as \(z \to 0\). Hence, (23) solves uniquely for \(z\) if \(\theta\) is not “too small.”\(^{30}\) This value of \(z\) can then be plugged into (21) to solve for \(p^{CM}\).

Notice that each asset’s CM subperiod price is tightly related to FM subperiod liquidity premia, \(\lambda(\cdot)\). Intuitively, an asset price equals its fundamental value—\(\beta\) for the government bond and zero for fiat money—plus any expected liquidity premia that cash and credit investors receive.\(^{31}\) For the government bond, cash investors receive a liquidity premium only in the state \(i = L\) because they repo finance all of their collateral holdings in the FM subperiod, while credit investors never receive a liquidity premium. Since only the cash investor receives a liquidity premium and only in state \(i = L\), the liquidity premium term \(\lambda(z + \bar{b})\) in (21) is multiplied by \(\pi L \sigma L\). For fiat money, cash investors receive liquidity premia in both states \(i = L, H\) while credit investors receive a liquidity premium in the state where the competitive repo price is less than 1, in state \(i = H\). Since only the cash investor receives a liquidity premium in the state \(i = L\), the liquidity premium term for state \(i = L\), \(\lambda(z + \bar{b})\), is multiplied by \(\pi L \sigma L\); since all investors receive a liquidity premium in state \(i = H\), the liquidity premium term for state \(i = H\), \(\lambda(z/\sigma H)\), is multiplied by \(\pi H (\sigma L + \sigma H) = \pi H\).

Since (steady state) equilibrium requires \(\phi_t(M_t + T^M_t) = \phi_{t+1}(M_{t+1} + T^M_{t+1})\), we have

\[
\frac{\phi_t}{\phi_{t+1}} = \frac{M_{t+1} + T^M_{t+1}}{M_t + T^M_t} = \frac{\mu_t}{M_t + T^M_t} = \mu_t.
\]

The central bank must set \(\mu_t = \pi^* + 1 \equiv \phi_t/\phi_{t+1}\) for all \(t\) to hit its inflation target of \(\pi^*\).\(^{32}\) The lump sum transfer in the CM subperiod of period \(t\), \(T^M_t\), required to hit the inflation target \(\pi^*\) is

\[
T^M_t = \pi^* M_t. \tag{25}
\]

Equations (21) and (23), along with \(\phi/\phi' \equiv \pi^* + 1\), can be used to pin down the equilibrium CM government bond price and real balances when the central bank repo facility is always

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\(^{30}\)If \(\theta\) is a very small number, then the benefit of an additional unit of liquidity to an investor is also very small. In this situation, the seller is basically the beneficiary of the additional liquidity because his bargaining power is high and the investor’s marginal benefit of the liquidity does not cover the marginal cost, \(\iota\). The investor’s bargaining power, \(\theta\), has to be sufficiently large to exceed the cost of accumulating any real balances.

\(^{31}\)Since sellers do not need liquidity in the DM subperiod, they do not attach a liquidity premium to these assets. As a result, sellers have no incentive to buy real balances or real assets in the CM subperiod—which we assumed—since these assets embed a liquidity premium in their CM subperiod prices. If sellers can participate in the FM subperiod, they would have no incentive to purchase the real asset in the CM subperiod since \(p^{FM}_i \leq p^{CM}_i\). Although the real asset can be purchased “cheap”—with real balances—in the FM subperiod in state \(i = H\), it is straightforward to show that sellers will not purchase real balances in the CM subperiod because the cost of holding real balances minus the expected benefit associated with purchasing the real asset in the FM subperiod in state \(i = H\) is strictly positive. Therefore, our assumption that sellers do not participate in the FM subperiod is not binding.

\(^{32}\)At this monetary growth rate, agents expect inflation to equal \(\pi^*\) in future periods.
inactive, which we denote as $\tilde{\rho}^{CM}$ and $\tilde{z}$, respectively. The other equilibrium variables of interest can be calculated as follows:

\[\tilde{\phi}_t = \frac{\tilde{z}}{([\pi^* + 1])^t M_0},\]
\[\tilde{\rho}^{FM}_H = \frac{1}{\lambda(\tilde{z}/\sigma_H) + 1}, \tilde{b}^*_H = \frac{(1 - \sigma_H)\tilde{z}}{\tilde{\rho}^{FM}_H \sigma_H}, \tilde{y}^*_H = Y(\frac{\tilde{z}}{\sigma_H}),\]
\[\tilde{\rho}^{FM}_L = 1, \tilde{b}^* L = \bar{b}, \tilde{y}^*_L = Y(\tilde{z} + \bar{b}),\]

where $M_0$ represents the nominal money stock at the beginning of date 0 and $T_0 = \pi^* M_0$.

This equilibrium is consistent with the stylized facts that as credit conditions tighten, aggregate output falls and liquidity becomes more scarce. To see this, define aggregate output in state $i$, $Q_i$, as

\[Q_i = \sigma_i f(\tilde{y}_i^*) + (1 - \sigma_i)f(\gamma^*).\]

Clearly, $Q_H < Q_L$ since $\tilde{y}_H^* < \tilde{y}_L^* < \gamma^*$ and $\sigma_H > \sigma_L$. Since credit conditions tighten when moving from state $i = L$ to state $i = H$, aggregate output declines as market liquidity becomes more scarce. Furthermore, when market liquidity is scarce, government bond prices become depressed, $\tilde{\rho}^{FM}_H < \tilde{\rho}^{FM}_L = 1$, which is consistent with the notion that when market liquidity “drys up” assets sell at “fire sale” prices, i.e., the repo price is less than 1, the fundamental value of a one-period asset at the end of one period.

4 An active central bank repo facility in state $H$

Since market liquidity is scarce in state $i = H$, there may be a role for the central bank to play as a liquidity provider. The central bank repo facility will be active in state $i = H$ only if its posted repo price is strictly greater than the equilibrium FM subperiod repo price when the facility is always inactive, i.e., if $p^{CB} > \tilde{\rho}^{FM}_H$. Furthermore, in any equilibrium where the central bank is active in state $i = H$, the competitive market repo price must equal the central bank’s posted repo price, i.e., $p^{FM}_H = p^{CB}$. If this was not the case then either: (i) $p^{CB} < p^{FM}_H = \tilde{\rho}^{FM}_H$ which implies that the central bank repo facility would be inactive, a contradiction; or (ii) $p^{CB} > p^{FM}_H$ which implies that cash investors’ demand for market (repo) liquidity would be zero and there will be an excess supply of repo finance in the FM subperiod competitive market. For the time being we assume that $p^{CB}$ is strictly greater than but close to $\tilde{\rho}^{FM}_H$. This assumption, which we will subsequently relax, implies that cash investors demand central bank repo finance in the FM subperiod but do not repo finance all of their collateral, i.e., $b^*_H < b$.

The characterization of equilibrium when the central bank repo facility is active in state $i = H$ is almost identical to that of an always inactive facility except that $p^{FM}_H$ is replaced with
p^{CB}$. Intuitively, in the previous section investors face prices \( \{ p^{FM}_L = 1, p^{FM}_H, p^{CM}, \phi/\phi_+ \} \) and now they face \( \{ p^{FM}_L = 1, p^{CB}, p^{CM}, \phi/\phi_+ \} \) since \( p^{CB} = p^{FM}_H \) in equilibrium. The asset pricing equations, which mimic (21) and (22), are

\[
p^{CM} = \beta + \beta \pi_L \sigma_L \lambda (z + \bar{b}) \tag{26}
\]

and

\[
\iota = \pi_L \sigma_L \lambda (z + \bar{b}) + \pi_H \lambda (z + p^{CB} b^*_H). \tag{27}
\]

Comparing these with (21) and (23), a notable and important quantitative difference between worlds with and without an active central bank repo facility lies in the cash investor’s state \( i = H \) real balance holdings. When the standing repo facility is always inactive, the cash investor’s real balances at the beginning of the DM subperiod in state \( i = H \) are \( z + p^{FM}_H b^*_H = z/\sigma_H \); when it is active in state \( i = H \), real balance holdings are \( z + p^{CB} b^*_H = z + p^{CB} (b^*_H + b^{CB}_H) = z/\sigma_H + p^{CB} b^{CB}_H \).

Intuitively, the standing repo facility allows cash investors to augment their real balance holdings beyond what is supplied in private markets. To complete the characterization of the equilibrium, we need an updated version of (8) with equality for \( i = H \). In particular, if we replace \( b^*_i \) with \( b^*_H \) and \( p^{FM}_H \) with \( p^{CB} \) in (8) we get

\[
\lambda (z + p^{CB} b^*_H) = 1 - p^{CB}/p^{CB}. \tag{28}
\]

The equilibrium can be solved sequentially. First, substitute (28) into (27) to get

\[
\iota = \pi_L \sigma_L \lambda (z + \bar{b}) + \pi_H \frac{1 - p^{CB}}{p^{CB}}, \tag{29}
\]

which solves the equilibrium \( z \).\(^{33}\) Second, substitute the equilibrium \( z \) into (26) and use this equation to solve for equilibrium \( p^{CM} \).\(^{34}\) We use the ‘hat’ accent to denote the equilibrium objects under \( p^{CB} \geq \tilde{p}^{FM}_H \). For example, \( \tilde{p}^{CM} \) and \( \tilde{z} \) denote the equilibrium CM subperiod bond price and real balances, respectively. By construction, \( \tilde{p}^{CM} = \tilde{p}^{CM} \) when \( p^{CB} = \tilde{p}^{FM}_H \) which implies that an increase in \( p^{CB} \) from \( \tilde{p}^{FM}_H \) necessarily decreases \( \tilde{z} \) from \( \tilde{z} \)—see (29)—and increases \( \tilde{p}^{CM} \) from \( \tilde{p}^{CM} \)—see (26). Intuitively, an active central bank repo facility makes government bonds more liquid resulting in a higher bond premium, \( \tilde{p}^{CM} - \beta \), as reflected by its higher CM subperiod price. And because real balances are costly to hold, an increase in the liquidity of

\(^{33}\) Again, for existence of equilibrium \( \theta \) cannot be too small.

\(^{34}\) Recall that \( \iota = (1 + \pi^*)/\beta - 1 \) can be viewed as being “exogenous.” Below, we discuss how the central bank is able to hit the inflation target \( \pi^* \).
bonds induces investors to reduce their real balance holdings. After obtaining \( \hat{z} \), we can obtain \( \hat{b}_H \) from (28). Other equilibrium objects of interest are listed as below:

\[
\hat{p}_{FM}^H = p^{CB}, \quad \hat{b}_H^C = \frac{(1 - \sigma_H)}{p^{CB}\sigma_H}, \quad \hat{b}_H^C = \hat{b}_H - \hat{b}_H^C, \quad \hat{y}_H = Y(\hat{z} + p^{CB}\hat{b}_H^C), \quad \hat{p}_{FM}^L = 1, \quad \hat{b}_L^C = \bar{b}, \quad \hat{y}_L = Y(\hat{z} + \bar{b}).
\]

Since the central bank repo facility is inactive in state \( i = L \), the CM subperiod transfer \( T_t^M \) given by (25) is consistent with the central bank hitting its inflation target. In state \( i = H \), however, (25) is not consistent with the central bank hitting its inflation target, \( \pi^* \). Intuitively, the central bank earns interest income from its repo transaction in state \( i = H \): the money balances it receives from cash investors who settle their repo obligations in the CM subperiod exceeds the money balances they provide to the cash investors in the FM subperiod. Therefore, if state \( H \) occurs in period \( t \) and the transfer is given by (25), then \( M_{t+1} < M_t(1 + \pi^*) \). This necessarily implies that if the central bank gives a transfer equal to (25) in all states \( i = H, L \), then investors’ inflation expectations must necessarily be strictly less than \( \pi^* \) and, as a result, the central bank will not hit its inflation target \( \pi^* \).\(^{35}\) In order to validate inflation expectations of \( \pi^* \), the central bank must increase its CM subperiod transfer beyond (25) by an amount \((1 - p^{CB})b_H^*/\phi_H \) in state \( i = H \), which represents the interest income earned by the central bank’s repo facility.\(^{36}\) When the central bank rebates the repo interest income back to the economy, the aggregate money supply growth rate in all states will be equal to \( \pi^* \). Here are two important observations. First, there is no inconsistency between a central bank achieving its long run inflation target while, at the same time, providing liquidity to financial markets when market liquidity is scarce, a point made by the Thornton and Bagehot. Second, the above results and discussion clearly indicate that the central bank’s role of lender of last resort—via standing repo—is a purely monetary/liquidity operation that has no fiscal policy implications.

It may seem puzzling at first that an injection of nominal money balances in the FM subperiod has real effects in an economy where prices are flexible. This puzzle can be resolved when it is recognized that the increased nominal balances are withdrawn later on in the period—in the CM subperiod—when cash investors repurchase their collateral from the central bank repo

\(^{35}\)More specifically, the standing repo facility injects \( p^{CB}b_H^*/\phi_H \) nominal balances in the FM subperiod and “withdraws” \( p^{CM}b_H^*/\phi_H \) nominal balances when buyers repurchase their collateral from the standing repo facility, where \( \phi_H = \phi_{t-1}/(1 + \pi^*) \). Since \( p^{CB} < p^{CM} \), if the CM subperiod transfer is equal to \( \pi^* M_t \), then \( M_{t+1} < M_t(1 + \pi^*) \).

\(^{36}\)That is, the total transfer in state \( i = H \) must be \( T_t^M = \pi^* M_t + (p^{CB} - p^{CM})b_H^*/\phi_H \). Notice that by construction, we have \( M_{t+1} = M_t + (p^{CB}b_H^* - p^{CM}b_H^* + T_t^M)/\phi_H = (1 + \pi^*)M_t \) in state \( H \).
facility. The net result (along with appropriate CM subperiod transfers \(T_t^M\)) is that inflation expectations—equal to \(\pi^*\)—will be unaffected. Hence, a nominal FM subperiod injection of money balances via the repo facility translates into higher DM subperiod real balances and, therefore, a higher transfer of investment goods between cash investors and sellers, and higher output in the CM subperiod.

Although the central bank’s repo facility enhances the liquidity of government bonds, it is not obvious that investors and sellers are better off because of it. We show below that total consumption increases in state \(i = H\) and decreases in state \(i = L\),\(^{37}\) which implies that the effect on social welfare is ambiguous. We assess the effect that the central bank repo facility has on the economy by appealing to a measure of social welfare that sums the discounted expected utility of agents in the economy, where the planner discounts the future at the same rate as agents. Owing to the linearity of the CM subperiod utility functions, this measure of social welfare simplifies to the difference between the discounted sum of per period expected investment output—which also equals discounted sum of per period expected consumption of investors and sellers in the CM subperiod—and the discounted sum of expected cost associated with producing the investment good by sellers in the DM subperiod. Since we focus on steady-state equilibria, our measure of social welfare is proportional to \(W(p^{CB})\) plus a constant, where

\[
W(p^{CB}) \equiv \pi_H \sigma_H [f(\hat{y}_H^c)] - c(\hat{y}_H^c) + \pi_L \sigma_L [f(\hat{y}_L^c)] - c(\hat{y}_L^c)
\]

and \(\hat{y}_H^c\) and \(\hat{y}_L^c\) depend on \(p^{CB}\) through the equilibrium conditions. The function \(W(p^{CB})\) captures the “welfare” generated by the cash investors; the welfare generated by credit investors is independent of \(p^{CB}\) and hence is a constant.

The model and analysis allow for the possibility of no aggregate risk: this happens whenever \(\pi_L = 0\) or \(\pi_H = 0\). Our first main result provides an important insight about central bank repo facilities when there is no aggregate risk.

**Proposition 1** When there is no aggregate risk in the economy, an active central bank repo facility cannot increase welfare.

See Appendix A for the proof.

One might think that since an active central bank repo facility provides cash investors with additional cash from their collateral holdings, they will be able to increase their investment

\(^{37}\)We demonstrate this in Proposition 2.
purchases and production of the consumption good and, as a result, social welfare increases. This intuition is incorrect. Because real balances are costly to hold, investors reduce their money accumulation, \( z \), in the CM subperiod by the amount of liquidity that is provided by the central bank repo facility. This result is reminiscent of Holmstrom and Tirole (1998), where government provided liquidity cannot improve outcomes in the absence of aggregate risk. In their model, there is enough private liquidity to finance all borrowing needs and government provided liquidity—that is provided by government debt—is not needed. In our model, however, government provided liquidity crowds out, one for one, market liquidity by decreasing the amount of real balances that investors demand.

When there is aggregate risk, a central bank repo facility can improve matters for society under the conditions described in the following proposition,

**Proposition 2**

(i) An active central bank repo facility increases investment and consumption in state \( i = H \) and decreases both in state \( i = L \). (ii) If \( \lambda / \lambda' \) is increasing on \([\tilde{z} / \sigma_H, \infty)\), then an active central bank repo facility can increase social welfare when \( \sigma_H \) is “sufficiently large” in the liquidity scarce state \( i = H \).

See Appendix B for the proof.

Just as in Proposition 1, an active central bank repo facility in state \( i = H \) reduces investors’ demand for real balances in the CM subperiod. The reduced real balance holdings, \( z \), necessarily implies that investment and consumption for cash investors and sellers fall in state \( i = L \) because total liquidity of a cash investors, \( z + p^F_M b^*_L = z + \hat{b} \) falls. The benefit associated with the central bank repo facility is the increased investment and consumption that occurs in state \( i = H \). The requirement that \( \lambda / \lambda' \) is increasing on \([\tilde{z} / \sigma_H, \infty)\) has a nice economic interpretation and is not very restrictive.\(^{38}\) The ratio \( \lambda / \lambda' \) is more likely to be increasing on \([\tilde{z} / \sigma_H, \infty)\), the more concave is \( f(\cdot) \) and the more concave is \( f(\cdot) \), the faster the marginal value of liquidity increases as

\[ \frac{d[\lambda(x)/\lambda'(x)]}{dx} = \frac{1}{\gamma}[(\gamma + 1)(x + \varepsilon)^{\gamma} - 1] = \frac{(x + \varepsilon)^{\gamma}}{\gamma}[(\gamma + 1) - (x + \varepsilon)^{-\gamma}]. \]  

In state \( i = H \), \( x \equiv \tilde{z} + \tilde{p}^F_M b^* = \tilde{z} / \sigma_H \) meaning that \( \lambda / \lambda' \) is increasing on \([\tilde{z} / \sigma_H, \infty)\) if \( (\tilde{z} / \sigma_H + \varepsilon)^{-\gamma} - 1 < \gamma \). Because \( (\tilde{z} / \sigma_H + \varepsilon)^{-\gamma} - 1 = \lambda(\tilde{z} / \sigma_H) \), we can use (28) to deduce that \( (\tilde{z} / \sigma_H + \varepsilon)^{-\gamma} - 1 < \gamma \) iff

\[ r^F_M < \gamma, \]

where \( r^F_M = \tilde{p}^CM / \tilde{p}^FM - 1 \) is the FM subperiod competitive repo rate when the central bank does not operate a standing repo facility. If \( \gamma \) is extremely low, say 0.1, a standing repo facility will be welfare improving if \( r^F_M < 10.0\% \). Even in this example the real rate has to be unrealistically high in order for \( \lambda / \lambda' \) to be decreasing on \([\tilde{z} / \sigma_H, \infty)\).
liquidity becomes scarce.\footnote{Notice that an increasing $\lambda / |\lambda'|$ implies that $\lambda / |\lambda'|$ is decreasing because $\lambda' < 0$. The latter means that as liquidity increases, the marginal value of liquidity normalized by $|\lambda'|$ decreases. Because $|\lambda'|$ is normally decreasing, this restriction requires that the marginal value of liquidity decreases sufficiently fast.} Hence, Proposition 2(ii) essentially says that an active central bank repo facility will be welfare enhancing if the marginal value of liquidity is significantly higher in the liquidity scarce state $i = H$ than the liquidity abundant state $i = L$ or, equivalently, if $\sigma_H$ is sufficiently large.\footnote{An alternative interpretation is that of a semi-elasticity; how much real balances $z$ change for a one percentage change in the cost of liquidity.}

We now provide some numerical examples that illustrate what is meant by $\sigma_H$ being “sufficiently large” and when the central bank provision of liquidity through repo finance is beneficial to society. The various panels in Figure 2 show how social welfare changes with the central bank’s repo bond price $p^{CB}$ for different values of $\sigma_H$, where $\sigma_H = 0.87, 0.95$ and $0.80$ in the left, middle and right panels, respectively. In all cases, state $i = H$ is characterized by scarce market liquidity when the central bank repo facility is always inactive.\footnote{The values for the other parameters and functional forms in the examples are: $\beta = 0.98, \theta = 0.7, \gamma = 0.7, \varepsilon = 0.05, \iota = 0.04, \bar{b} = 0.2, \pi_L = 0.8, \pi_H = 0.2, \sigma_L = 0.03, \epsilon(y) = y$ and $f(y) = \frac{(y + \varepsilon)^{1-\gamma} - \varepsilon^{1-\gamma}}{1-\gamma}$.} We plot the percentage change in social welfare compared to the case when the central bank’s repo facility is always inactive. In the left panel of Figure 2, social welfare, at least initially, smoothly increases and then decreases with $p^{CB}$. But as $p^{CB}$ continues to increase, the social welfare function kinks at $p^{CB} = \bar{p}^{CB}$, where the qualitative nature of the equilibrium changes. At and beyond the kink, the equilibrium is characterized by cash investors repo financing all of their collateral and credit

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Numerical Examples: Welfare}
\end{figure}
investors supplying all of their real balances for repo finance in the state $i = H$. Notice that the social welfare maximizing repo price $p^{CB}$ lies in between $\hat{p}_H^{FM}$ and $\bar{p}^{CB}$ in the left panel, which implies that cash investors do not repo finance all their collateral in the FM subperiod, i.e., $\hat{b}_H + \hat{b}_H^{CB} < b$. Hence, total liquidity remains scarce in state $i = H$ at the social welfare maximizing central bank repo price. In contrast, social welfare reaches its maximum at $\bar{p}^{CB}$ in the middle panel. Here, market liquidity is so scarce—since $\sigma_H = 0.95$—that buyers want to repo finance all of their collateral at the social welfare maximizing repo price, $\bar{p}^{CB}$. When $\hat{p}_H^{FM} < p^{CB} < \bar{p}^{CB}$, however, buyers do not repo finance all of their collateral and total liquidity is scarce in state $i = H$. The left and middle panels provide examples of $\sigma_H$ being “sufficiently large” and, as a result, an active central bank repo facility in state $i = H$ is welfare enhancing.

The right panel in Figure 2 provides an example where $\sigma_H$ is not “sufficiently large.” When $\sigma_H = 0.80$, market liquidity is scarce in state $i = H$ in the equilibrium when the central bank’s repo facility is always inactive. But unlike the two other examples, $\sigma_H$ is not sufficiently large. When the central bank’s repo facility becomes active in state $i = H$ (by having $p^{CB}$ increase above $\hat{p}_H^{FM}$), social welfare immediately declines—see the right panel. In this example, expected market liquidity is impaired by an active central bank repo facility because the decrease in market liquidity in state $i = L$ dominates the increase in total liquidity in state $i = H$.

These examples indicate that a central bank repo facility is beneficial only when market liquidity is “really” scarce in state $i = H$ and that scarce market liquidity is not a sufficient reason for a central bank to provide liquidity support via its repo facility. Our two-state model succinctly identifies the costs and benefits associated with an active standing repo facility when state $i = L$ is characterized by an abundance of market liquidity and $i = H$ by scarcity. Although the two-state model is both simple and illustrative, it does not capture all of the potential equilibrium configurations that can arise, and the equilibria that do arise seem to depend critically on how we choose values for $\sigma_L$ and $\sigma_H$. To remedy these issues we expand the number of possible states from 2 to $[0, 1]$. When $\sigma \in [0, 1]$, all possible configurations will arise in equilibrium.

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42This equilibrium configuration can exist in the case of an always inactive central bank repo facility when the value of $\sigma_H$ greater than $\sigma_L$ but is not “too large.” Such an equilibrium configuration, when the central bank is always inactive, is characterized by scarcity in state $i = L$ and by having cash investors repo finance all of their collateral and credit investors supply all of their real balances for repo finance in state $i = H$. In our more general model, in Section 5 and Appendix D, we fully characterize such an FM subperiod equilibrium outcome.

43For example, a two-state model cannot have FM subperiod equilibria that are characterized by market liquidity that is scarce, abundant and sufficient. By construction, only two of the three possible configurations can arise.
5 A General Model

We now assume that $\sigma$ is continuous and independently and identically distributed over $[0, 1]$ with distribution $F(\sigma)$. Three distinct equilibrium configurations can arise in the FM subperiod when the central bank repo facility is always inactive: when $\sigma$ is "small," the equilibrium is characterized by abundant market liquidity; when $\sigma$ is "large," it is characterized by scarce market liquidity; and when $\sigma$ is neither small nor large, it is characterized by sufficient market liquidity, see figure 3.44

The critical cutoff values, $\tilde{\sigma}_L$ and $\tilde{\sigma}_H$, that separate the three equilibrium regions in Figure 3 are determined as follows.45 Market liquidity is abundant for all $\sigma$ that satisfy $\sigma \bar{b} < (1 - \sigma)z$. The critical cutoff $\tilde{\sigma}_L$ occurs when total market liquidity just becomes sufficient when the FM subperiod bond price is equal to 1, i.e., when $\tilde{\sigma}_L \bar{b} = (1 - \tilde{\sigma}_L)z$ or

$$\tilde{\sigma}_L = \frac{z}{z + \bar{b}}.$$  \hfill (32)

The critical cutoff $\tilde{\sigma}_H$ occurs when market liquidity is sufficient but becomes scarce when $\sigma$ is increased from $\tilde{\sigma}_H$, i.e., when $\lambda(z/\tilde{\sigma}_H) + 1 = 1/p_{\tilde{\sigma}_H}^{FM} = \tilde{\sigma}_H \bar{b}/(1 - \tilde{\sigma}_H)z$ or

$$\tilde{\sigma}_H = \frac{z[\lambda(z/\tilde{\sigma}_H) + 1]}{z[\lambda(z/\tilde{\sigma}_H) + 1] + \bar{b}}.$$  \hfill (33)

\footnote{We continue to assume that $\bar{b}$ is “not large” in the body of the paper. In Appendix E, we precisely characterize what it means for $\bar{b}$ to be “large” and “small,” as well the equilibria with and without an active standing repo facility when $\bar{b}$ is “large.”}

\footnote{The critical cutoff $\tilde{\sigma}_L$ is unique. The critical cutoff $\tilde{\sigma}_H$ need not be unique. We are, however, unable to generate any examples where $\tilde{\sigma}_H$ is not unique. We assume that the critical cutoff $\tilde{\sigma}_H$ is unique in the main body of the paper and provide conditions for a unique cutoff in Appendix C.}

\footnote{To understand this equality, notice from (8) we have that $\lambda(z/\tilde{\sigma}_L) + 1 > 1/p_{\tilde{\sigma}_L}^{FM} = 1$ since $z + \bar{b} = z/\tilde{\sigma}_L$ and $p_{\tilde{\sigma}_L}^{FM} = 1$. As $\sigma$ increases from $\tilde{\sigma}_L$, we have, by continuity,

$$\lambda(z/\sigma) + 1 > 1/p_{\sigma}^{FM} = \bar{b}\sigma/[(1 - \sigma)z]$$

for $\sigma$ “close” to but greater than $\tilde{\sigma}_L$ because $\sigma p_{\sigma}^{FM} \bar{b} = (1 - \sigma)z$ implies $p_{\sigma}^{FM} = (1 - \sigma)z/(\sigma \bar{b})$. Since

$$\lambda(z/\sigma) + 1 < \bar{b}\sigma/[(1 - \sigma)z]$$}

Figure 3: Market liquidity
When \( \sigma \) is continuous, both the bond pricing and money demand equilibrium equations are straightforward generalizations of (21) and (23), respectively. In particular, asset prices are simply weighted averages of the liquidity premia that arise across the various states \( \sigma \).\(^{47}\) For example, regarding real balances, cash investors have strictly positive liquidity premia in all states \( \sigma \) while credit investors have strictly positive premia only in states where market liquidity is not abundant, i.e., in states \( \sigma > \tilde{\sigma}_L \). And regarding government bonds, cash investors have strictly positive liquidity premia in states where they repo finance all of their collateral, i.e., in states \( \sigma \in (0, \tilde{\sigma}_H) \), while credit investors never receive a strictly positive liquidity premium.

We now show by way of numerical examples that the main results and insights from our two-state environment generally carry over to a continuous distribution world.\(^{48}\) When the central bank’s repo facility is always inactive the equilibrium CM subperiod asset price and real balance holdings for our parameterization are \( \tilde{p}^{\text{CM}} = 0.9990 \) and \( \tilde{z} = 0.9498 \), respectively.

Figure 4(a) illustrates how the equilibrium FM subperiod repo bond price, \( \tilde{p}^{\text{FM}}_\sigma \), varies with \( \sigma \) while figure 4(b) illustrates how the equilibrium quantity of collateral repo finance for the representative cash investor in the FM subperiod, \( \tilde{b}_c^\sigma \), varies with it. At lower values of \( \sigma \), market liquidity is abundant and the competitive repo price \( \tilde{p}^{\text{FM}}_\sigma \) equals 1, which is the payoff to the government bond in the subsequent CM subperiod. In these states, cash investors repo finance all of their collateral. Nevertheless, the demand for market liquidity relative to total potential supply of market liquidity is small because the total amount of collateral that is repo financed, \( \sigma \tilde{b} \), is relatively small. (The total supply of collateral that is repo financed is relatively small because \( \sigma \) is small.) The equilibrium has credit investors providing only a fraction of their money holdings in these \( \sigma \) states for repo finance. As \( \sigma \) increases, we enter the sufficient market liquidity region, \( \sigma \in (\tilde{\sigma}_L, \tilde{\sigma}_H) \), where the FM subperiod repo bond price \( \tilde{p}^{\text{FM}}_\sigma \) drops sharply from 1 with increases in \( \sigma \), see figure 4(a). In this region, cash investors repo finance all of their collateral and credit investors use all of their real balances for repo finance. The sharp decline in the FM subperiod bond price \( \tilde{p}^{\text{FM}}_\sigma \) as \( \sigma \) increases can be explained by the combination of a decline in total market liquidity—since \( 1 - \sigma \) falls—and an increase in total collateral supply—since \( \sigma \) increases and \( \tilde{b}_c^\sigma = \tilde{b} \). Finally, when \( \sigma \geq \tilde{\sigma}_H \), the FM subperiod repo bond price \( \tilde{p}^{\text{FM}}_\sigma \)

\[^{47}\text{The asset pricing equations are derived in Appendix D.}\]

\[^{48}\text{The parameters and functional forms are the same as described in footnote 41 and the only difference is the statistical properties of \( \sigma \). We now assume that the CDF of \( \sigma \), \( F \), is the standard uniform distribution with support concentrated on [0,1].}\]
continues to decline with increases in $\sigma$ but not as rapidly, as shown in figure 4(a). The critical value $\tilde{\sigma}_H$ occurs at the “second kink” in figure 4(a) in the repo bond price—where the slope changes from very steep to less steep—and at the kink in figure 4(b). In our example the region of sufficient market liquidity is very small; in particular, $(\tilde{\sigma}_L, \tilde{\sigma}_H) = (0.826, 0.835)$. For $\sigma > \tilde{\sigma}_H$, market liquidity is scarce and cash investors do not repo finance all of their collateral and finance less of their collateral as $\sigma$ increases, while credit investors supply all of their real balances for repo finance. This combination mitigates the decline in $\tilde{p}_F^{\sigma}$ as $\sigma$ increases compared to the previous region, as illustrated in figure 4(a). When market liquidity is scarce, the FM subperiod repo bond price is “too low” to induce cash investors to repo finance all of their collateral.

Figure 4(b) shows that the quantity of bonds that the typical cash investor repo finances is constant over the entire region of $\sigma$ where $\tilde{p}_F^{\sigma} = 1$ and market liquidity is abundant, as well as for the small region where $\tilde{p}_F^{\sigma} < 1$ and market liquidity is sufficient. Figure 4(c) illustrates the total money holdings of the cash investor after the FM subperiod transactions, $\tilde{z}_c^{\sigma} \equiv \tilde{z}_c + \tilde{p}_F^{\sigma} \tilde{b}_c^{\sigma}$. Money holdings are constant for all $\sigma$ where $\tilde{p}_F^{\sigma} = 1$, when market liquidity is abundant. When the FM subperiod repo bond price falls below 1—this happens when $\sigma > \tilde{\sigma}_L = 0.826$—the cash investor’s money holdings $\tilde{z}_c^{\sigma}$ decline with $\sigma$. But unlike the equilibrium FM subperiod repo bond price, $\tilde{p}_F^{\sigma}$, which initially declines rapidly and then tails off, the rate of decline in money holding appears to be more or less constant. This is because when $\tilde{p}_F^{\sigma}$ drops rapidly, cash investors continue to repo finance all of their collateral (when market liquidity is sufficient) and when the decline in $\tilde{p}_F^{\sigma}$ moderates, buyers are only repo financing a fraction of their collateral (when market liquidity is scarce). These effects work to smooth the decline in real balance holdings over the two regions where market liquidity is sufficient and scarce and $\tilde{p}_F^{\sigma} < 1$. 
We now examine the case where the central bank’s repo facility is sometimes active. Define \( \bar{\sigma} \) as the critical state such that for all \( \sigma > \bar{\sigma} \) cash investors obtain repo finance from both the competitive financial market and the central bank’s repo facility; and for all \( \sigma < \bar{\sigma} \) they only get it from the competitive market (because in these states the central bank repo facility is inactive). As above, we denote real balances, bond prices and other critical values for a central bank’s repo facility that is sometimes active (in states \( \sigma > \bar{\sigma} \)) with a “hat,” i.e., \( \hat{z}, \hat{p}_{CM}, \hat{p}_{FM}, \hat{\sigma}_L, \hat{\sigma}_H \) and so on. The precise relationship between \( \bar{\sigma} \) and the central bank’s repo price, \( p_{CB} \), depends on whether market liquidity is either scarce or sufficient at \( \bar{\sigma} \). If market liquidity is scarce at \( \bar{\sigma} \), then \( \bar{\sigma} > \hat{\sigma}_H \). This situation occurs if \( p_{CB} \) is not too big compared to \( \hat{p}_1^{FM} \), the competitive FM subperiod price when \( \sigma = 1 \) and the central bank repo facility is inactive. Then from (28) we have

\[
\lambda(\frac{z}{\bar{\sigma}}) = \frac{1 - p_{CB}}{p_{CB}}. \tag{34}
\]

In all states \( \sigma > \bar{\sigma} \), cash investors repo finance a total of \( b^* \) collateral using both financial markets and the central bank repo facility. In these states, market and central bank liquidity is scarce.\(^{49}\)

If market liquidity is sufficient at \( \bar{\sigma} \), then the critical value \( \hat{\sigma}_H \) does not exist. This situation occurs if \( p_{CB} \) is sufficiently big: cash investors will repo finance all of their collateral in states \( \sigma > \bar{\sigma} \). In fact, cash investor repo finance all of their collateral in all states of the world. The critical value \( \bar{\sigma} \) is determined by the equality of economy wide value of repo finance by cash investors, who each repo \( \bar{b} \) collateral at price \( p_{CB} \), with market liquidity, i.e., \( \bar{\sigma} p_{CB} \bar{b} = (1 - \bar{\sigma})z \) or

\[
\bar{\sigma} = \frac{z}{z + p_{CB} \bar{b}}.
\]

We now return to our numerical example to illustrate equilibrium outcomes and optimal central bank repo policy. Figure 5(a) illustrates the CM subperiod equilibrium bond price, \( \hat{p}_{CM} \), and real balance holdings, \( \hat{z} \), as functions of the central bank repo price \( p_{CB} \). When \( p_{CB} \) is strictly less than the FM subperiod bond price at \( \sigma = 1 \), i.e., when \( p_{CB} < \hat{p}_1^{FM} = 0.847 \), the central bank repo facility is always inactive and an increase in \( p_{CB} \) has no effect, at least initially, on the equilibrium outcomes. This is illustrated in Figure 5(a) by the horizontal blue and red lines for \( p_{CB} < 0.847 \). As \( p_{CB} \) increases beyond 0.847, cash investors will access the central bank’s repo

\(^{49}\)Market and central bank liquidity is scarce in the sense that cash investors do not sell all of their bond holdings in the FM subperiod and credit investors use all of their money balances to purchase bonds in the FM subperiod.
facility in an increasing number of $\sigma$-states to obtain additional liquidity.\textsuperscript{50} Because collateral (government bonds) can generate additional liquidity for cash investors, investors’ demand for government bonds in the CM subperiod increases. Hence, an increase in $p^{CB}$ increases the equilibrium bond price, $p^{CM}$, and decreases real balance holdings, $\hat{z}$, as illustrated by the blue and red lines, respectively, in Figure 5(a). The “kink” in the (red) money demand curve, which occurs at $p^{CB} = 0.923$, has special significance. For $p^{CB} < 0.923$, $\hat{\sigma}_H > \bar{\sigma}$ which means that states $\sigma < \bar{\sigma}$ are characterized by market and central bank liquidity scarcity.\textsuperscript{51} For $p^{CB} > 0.923$, $\hat{\sigma}_H$ does not exist, which means that for all $\sigma \geq \bar{\sigma}$ cash investors repo finance all of their collateral in the FM subperiod.\textsuperscript{52}

Figure 5(b) shows the percentage change in welfare for various values of $p^{CB}$. Welfare is maximized when the central bank sets $p^{CB} = 0.923$. Interestingly, this is the value of $p^{CB}$ at the kink of the money demand function in figure 5(a).\textsuperscript{53} The optimal central bank repo price has cash investors repo financing all of their collateral in all states of the world. When the central

\textsuperscript{50}When the central bank repo price initially increases beyond 0.847, the relevant bond price and money demand equations are given by (48) and (49), respectively, in Appendix D.

\textsuperscript{51}In this case, the equilibrium bond and money demand equations are given by (48) and (49), respectively, in Appendix D.

\textsuperscript{52}In this situation, the equilibrium bond and money demand function are given by (50) and (51), respectively, in Appendix D. In terms of our example, $p^{CB}$ is “large enough” when $p^{CB} \geq 0.923$ and, as a result, $\hat{\sigma}_H$ does not exist.

\textsuperscript{53}We parameterize the example in the subsequent section so that $\hat{\sigma}_H > \bar{\sigma}$. We were unable to generate any examples where the pdf is symmetric on $[0, 1]$ and $\hat{\sigma}_H > \bar{\sigma}$. We are able to generate an example characterized by $\hat{\sigma}_H > \bar{\sigma}$—in the next section—by assuming a non-symmetric pdf.
Figure 6: Financial Market with CB Intervention

bank chooses the optimal repo price, welfare increases by about 5.93 basis points compared to an always inactive standing repo facility and real balance holdings decline by about 3.5%, (more specifically, $\hat{\Delta} = 0.9171$ versus $\tilde{\Delta} = 0.9498$, respectively).

Figure 6 compares financial market outcomes for central bank repo facility that chooses the optimal repo price—in red—and an always inactive facility—in blue.\(^{54}\) As can be seen in Figure 6(a), the optimal central bank repo rate provides a cap on market rates in high $\sigma$-states of the world.\(^{55}\) When the central bank repo facility is always inactive, scarcity of market liquidity becomes more acute in higher $\sigma$-states and as $\sigma$ increases, higher market repo rates follow.

\(^{54}\) We discuss the yellow lines below.

\(^{55}\) Using market and central bank repo rates instead of FM subperiod repo bond prices, help facilitate the comparison between an active and always inactive central bank repo facility.
When market liquidity is no longer abundant the market repo rate initially and significantly spikes from zero—for both the optimal and the always inactive central bank repo facilities—since cash investors continue to repo finance all of their collateral even as repo rates rise (or, equivalently, FM subperiod bond prices fall). For an always inactive central bank repo facility, the increase in the repo rate is moderated as $\sigma$ increases since cash investors choose to repo finance smaller fractions of their collateral holdings (owing to lower FM subperiod market repo bond prices). At the point where market liquidity is no longer abundant—and repo rates exceed zero—the market repo rate for an always inactive facility is, at least initially, less than the market repo rate associated with an optimally set central bank repo rate, i.e., the red line lies above the blue.\textsuperscript{56} But as $\sigma$ continues to increase at some point—where the blue and red lines cross in figure 6(a)—the repo rate associated with the optimal repo facility is always less than the market repo rate for an always inactive facility. These observations highlight the role played by a central bank repo facility: the facility essentially insures investors against aggregate liquidity risk by equalizing marginal liquidity costs across very high-$\sigma$ states of the world.

If the central bank repo facility is always inactive, then market liquidity will be scarce in high-$\sigma$ states. In this situation as $\sigma$ increases, FM subperiod repo bond prices continue to fall (and market repo rates continue to increase); cash investors respond to these lower repo bond prices (higher market repo rates) by repo financing less collateral. As a result, total liquidity for a representative cash investor, $\tilde{z}_c = \tilde{z} + \tilde{p}_{FMb}$, strictly decreases as $\sigma$ increases, as illustrated by the blue line in Figure 6(b). If, instead, when the central bank chooses the optimal repo price (rate), the facility will become active when market liquidity become just sufficient and in higher $\sigma$ states.\textsuperscript{57} Hence, the collateral that the representative cash investor repo finances is constant over all $\sigma$ and equal to $\bar{b}$. As a result, total real balance holdings for the representative cash investor entering the DM subperiod, $\check{z}_c^\sigma = \check{z} + \check{p}_{FMb}$, are constant across all the high $\sigma$-states, where $\sigma \geq \check{\sigma}$, as illustrated by the red line in Figure 6(b). Although cash investors repo finance all of their collateral in the FM subperiod, total liquidity in higher $\sigma \geq \check{\sigma}$ states is less than in lower $\sigma$ states because the FM subperiod repo bond prices are lower in the former. When comparing the optimal and the always inactive standing repo facilities, there exists a trade-off in the investor’s CM subperiod real balance holdings. An investor’s CM subperiod real balance holdings are higher when the facility is always inactive compared to the optimal repo facility.

\textsuperscript{56} This result should be anticipated since $\check{z} > \tilde{z}$.

\textsuperscript{57} We emphasize that for this particular example is characterized by sufficient market liquidity at $\sigma = \check{\sigma}$. 
When the facility is always inactive the investor essentially “self insures” against high $\sigma$’s by accumulating more real balances in the CM subperiod compared to an investor that has the optimal standing repo facility insuring against scarcity. This difference in real balance holdings can be seen in figure 6(b) and is measured by the vertical distance between the blue and red lines when market liquidity is abundant, i.e., at lower values of $\sigma$. This observation is rather important when assessing the potential welfare gains associated with an active central bank repo facility. In particular, investment and consumption will be lower in low $\sigma$-states and higher in high $\sigma$-states for the optimal central bank repo facility compared to one that is always inactive.

When the central bank repo rate is optimally set, the central bank repo finances an ever increasing amount of collateral when $\sigma$ is high and increasing as illustrated by the red line in figure 6(c). The total value of repo finance in the FM subperiod, including central bank repo finance, is illustrated in Figure 6(d). Here the red line shows that the total value of repo finance declines as $\sigma$ increases when market liquidity is sufficient and the central bank repo facility remains inactive. Over this range of $\sigma$ although more cash investors are repo financing $\bar{b}$ collateral as $\sigma$ increases, the decline in repo bond prices results in a reduction in the total value of market value of repo finance. When the central bank repo facility becomes active at $\sigma = \bar{\sigma}$, further increases in $\sigma$ results in higher total value of repo finance since the central bank’s repo facility stabilizes the repo bond price at $p^{CB}$. When the standing repo facility is always inactive—given by the blue line in Figure 6(d)—once market liquidity ceases to be abundant, i.e., $\sigma > \hat{\sigma}_L$, the total value of repo finance declines to zero as $\sigma$ increases.

In our example, the optimal repo rate for central bank repo facility is 8.37% which implies that 82.08% of the time market liquidity is abundant with an associated market repo rate equal to 0.0%. In an sense, the central bank’s repo rate can be interpreted as a penalty rate in the sense of Bagehot (1973) since, ex ante, it will exceed the equilibrium market rate most of the time. Cash investors use the standing repo facility in states $\sigma \geq 0.8323$, which occurs only 16.77% of the time: hence, the vast majority of the time, the standing repo facility is not used. When the central bank repo facility is always inactive the market repo rate can be as high as 18.07%. Nevertheless, even in this environment the market repo rate will be zero most (82.60%) of the time. These numbers imply that the ex ante welfare gain associated with a central bank repo facility will be very small since the facility is seldom used: the welfare gain associated with an optimal central bank repo facility is about 6 basis points. However, and importantly, when cash investors do use the facility, the state contingent welfare gain can be quite substantial. For
example, when $\sigma = 0.9$, the welfare gain is 0.57% and when $\sigma = 0.95$, it is 1.58%.

The yellow lines in Figure 6 describe a situation where the central bank’s repo rate is set at a non-optimal and very low rate, equal to 0%, the rate that prevails most of the time in our previous examples. Figures 6(b) and 6(d) indicate that setting such a low rate generates stable outcomes across states. In particular, the liquidity that a representative cash investor brings into the DM subperiod is almost invariant to the value of $\sigma$—see figure 6(b)—and the total value of repo finance is essentially an increasing function of $\sigma$—see figure 6(d). Figure 6(c) illustrates that central bank repo finance is always higher when the repo rate is set lower than the optimal setting, and that the facility will be used for lower values of $\sigma$. Although these outcomes are stable across states, consumption for cash investors will be lower in most states compared to an optimally set central bank repo rate, see Figure 6(b). Since investors know they have access to “cheap” liquidity in the FM subperiod, they accumulate less real balances in the CM subperiod and, as a result, consume less in most states of the world. This results in a lower level of welfare compared to the optimal setting. In fact, for this example, the 0% central bank repo rate results in a level of welfare that is lower than that associated with an always inactive central bank repo facility.

An important insight from our benchmark, two-state model is that market liquidity has to be very scarce if a central bank standing repo facility is to be beneficial. This insight carries over to the continuous state case. When $\sigma \in (0, 1)$, by construction, there always exist states of the world where market liquidity is very scarce. As a result a central bank repo facility can always be welfare improving. But providing central bank repo liquidity “too generously”—i.e., when liquidity is scarce but not very scarce, as in the yellow lines in Figure 6—can actually result in lower welfare compared to the central bank never providing repo finance. When the repo rate is optimally set, financing with central bank repo will be a low probability event, meaning that even though a central bank repo facility is beneficial and state contingent gains may be large, the \textit{ex ante} welfare gain will typically be very small. Baghot’s prescription to “lend freely on good capital but at a high rate” is captured in all these examples. Setting a central bank repo rate that is “too low” creates a moral hazard problem—investors’ accumulate too little private liquidity since they can rely on the central bank for cheap liquidity—which results in a decrease in social welfare.
6 ‘Standing’ v. ‘Emergency’ Central Bank Repo Facilities

The Federal Reserve established a *standing* repo facility in July 2021. This facility is available every business day to accredited counterparties and the facility’s repo rate (price) and other policy parameters are both known and clearly specified in advance. We will use the term ‘standing facility’ to describe Bagehot’s policy prescription that a central bank’s liquidity lending rates, as well as other important parameters, be clearly articulated and known to all. Prior to July 2021, the Federal Reserve supplied liquidity to financial markets in the aftermaths of major liquidity events on an *ad hoc* basis, through emergency or temporary repo operations and facilities. For example, after the onset of the 2007-08 financial crisis, the Fed established a variety of emergency repo-type facilities and operations, such as the Primary Dealer Credit Facility. More recently, when a shortfall in central bank reserves significantly and suddenly elevated overnight rates in September 2019, the Fed offered up to $75 billion in overnight repo funding using an auction mechanism with minimum reserve bid. And in March 2020, in response to a huge increase in demand for US dollars resulting from the COVID-19 pandemic, which resulted in dramatic spikes in market repo rates, the Federal Reserve deployed a number of overnight repo operations and facilities, including a Primary Dealer Credit Facility, to supply liquidity to domestic and international market participants. We will use the term ‘emergency facility’ to describe earlier Federal Reserve *ad hoc* facilities.

In this section we ask whether a standing repo facility—as described by, say, the Federal Reserve’s current standing repo facility—provides superior outcomes to an emergency repo facility—as described by, say, past Federal Reserve interventions where policies arise after a major liquidity shock has been realized. Even though a standing repo liquidity facility can insure all investors against extreme market liquidity scarcity events, it is not obvious that it is better than an emergency repo facility. There is a tradeoff. Although a standing facility insures all investors against adverse shocks, it also reduces their incentive to accumulate CM subperiod real balance holdings—and, hence, reduces market liquidity—compared to an emergency facility. This implies that investment and consumption levels associated with a standing repo facility will be less than those associated with an emergency facility in states of the world where investors do not obtain any central bank repo finance (because market liquidity is either abundant and/or sufficient). The emergency repo facility will generate higher consumption and investment, com-

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58 The Primary Dealer Credit Facility provided overnight secured (repo) loans to primary dealers.
pared to a standing facility, the vast majority of the time since the central bank’s repo facility is not accessed the vast majority of the time.

We view the timing in Figure 1 as one that describes a standing repo facility because the facility is always available to all investors at clearly announced FM subperiod terms of trade. We interpret an emergency facility as one where investors understand that the central bank will intervene in the event of severe scarcity of market liquidity but only after it observes the scarcity. But to observe scarce market liquidity necessarily implies that (some) investors trade in the FM subperiod competitive financial market without the support of the central bank. If the competitive financial market reveals significant scarcity of market liquidity, then the central bank reacts by providing liquidity via a repo facility to remaining investors. We model the notion of an emergency repo facility by subdividing the FM subperiod into two parts—“early” and “late”—and assigning a fraction of investors to visit the FM subperiod early and the remainder late. More specifically, at the beginning of the period when investors learn whether they are cash or credit investors, they also learn whether they participate in the early or late FM subperiod. A fraction $\alpha$ of investors enter and trade in the “early” FM subperiod—without the aid of any central bank repo facility—and then exit. The central bank and the remaining $1 - \alpha$ investors observe the market repo price/rate generated by the $\alpha$ investors in the early FM subperiod. The central bank establishes an emergency repo facility if and only if the early FM subperiod repo price falls below a predetermined level $p_{CB, \alpha}^{E}$. After the central bank’s decision to establish (or not) an emergency facility, the remaining $1 - \alpha$ investors enter the late FM subperiod. These investors trade in a late FM subperiod competitive financial market and with the central bank, if an emergency facility was established. Intuitively, this timing, which is illustrated in Figure 7, operationalizes the idea that of an ad hoc central bank repo facility, i.e., a repo facility that is established only after significant scarcity in market liquidity is actually observed.

As above, the repo price $p_{CB}^{E}$ will be associated with a state $\sigma_\alpha$ such that a central bank emergency facility with repo price $p_{CB}^{E}$ is established in all states $\sigma > \hat{\sigma}_\alpha$. Just as in previous sections, both the equilibrium bond pricing and money demand equations are functions of the expected liquidity premia they generate. The money demand and bond pricing equations not

\[\text{\footnotesize 59} \text{Since the central bank can infer } \sigma \text{ after } \alpha \text{ investors trade in the early FM competitive market, an alternative model of an emergency repo facility is to have a state-contingent central bank repo price, i.e., } p_{CB}^{E}(\sigma). \text{ That is, after } \sigma \text{ is “observed” after the early FM subperiod, the central bank can post a repo price that varies with } \sigma.\]

\[\text{\footnotesize 60} \text{In all states } \sigma > \hat{\sigma}_\alpha, \text{ the early market repo price is strictly less than } p_{CB}^{E}. \text{ When there is an emergency facility, critical parameters will be indexed by } \alpha, \text{ e.g., } \sigma_\alpha, \hat{\sigma}_L, \hat{\sigma}_H \text{ and so on.}\]

\[\text{\footnotesize 61} \text{The precise form of the equilibrium bond pricing and money demand equations for the emergency repo facility are derived and can be found in Appendix D.}\]
only reflect the existence of an emergency repo facility in states $\sigma > \tilde{\sigma}_\alpha$ but also that $\alpha$ cash investors—the early investors—cannot access a central bank repo facility in the FM subperiod in those states. Let $\hat{\sigma}_H^\alpha$ be the cut-off that separates the region of sufficient liquidity and the region of scarce liquidity in the early FM subperiod. It always exists if $\alpha > 0$. The precise form of the bond pricing and money demand equations depend, in part, on whether $\tilde{\sigma}_\alpha$ is larger than $\hat{\sigma}_H^\alpha$.

The equilibrium bond pricing and money demand equations are functions of $\alpha$, the measure of investors that arrive in the early FM subperiod. Increasing $\alpha$ results in higher consumption in all states $\sigma < \tilde{\sigma}_\alpha$. Intuitively, increasing $\alpha$ increases the probability that cash investors will not be able to access central bank liquidity when $\sigma > \tilde{\sigma}_\alpha$. Because of this, investors effectively self-insure against this possibility by increasing their real balance holdings in the CM subperiod. As a result, consumption increases in states $\sigma < \tilde{\sigma}_\alpha$. Holding all else constant, this increases welfare. However, increasing $\alpha$ decreases consumption for a greater measure of early cash investors in states $\sigma > \tilde{\sigma}_\alpha$ compared to the consumption of late cash investors and, holding all else constant, this decreases welfare. The total effect on welfare from a change in $\alpha$ is ambiguous. To understand how changes in $\alpha$ affects welfare we undertake the following quantitative exercise.

For this exercise, we set $f(y) = y^{1-\gamma}$, $c(y) = y$ and $F$ to a beta distribution with parameters $a_1$ and $a_2$. We need to provide values for parameters $a_1$, $a_2$, $\tilde{b}$, $\beta$, $\gamma$, $\mu$ and $\theta$. Some of
<table>
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<tr>
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<td>Bond Quantity</td>
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<td>2.08% Average BGCR</td>
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</table>

Table 1: Benchmark Parameters for the Quantitative Analysis

these parameter values can be determined directly from data while other values can come from predictions of our model that are disciplined by or match the data. We consider an annual model so we set \(\beta = 0.98\). We set \(1 - \gamma = 0.4\) which can be thought of as matching the capital share of 40% found in the data under the assumption that labor supply is inelastic. We set \(\mu = 0.02\) to generate a 2% annual inflation and \(\theta = 0.5\) so that the investors and the investment good producers have equal bargaining power. It is difficult to pick \(a_1\) and \(a_2\). As a benchmark, we set \(a_1 = 2\) and \(a_2 = 4\), which implies that the pdf is skewed to the left. One potential benefit of this choice of \(a_1\) and \(a_2\) is that extreme scarcity in market liquidity can be interpreted as a tail event.\(^{62}\) Given these parameters, we set \(\bar{b}\) such that the average repo rate from the model matches the average Broad General Collateral Rate (BGCR) of 2018 and 2019, which is 2.08%. The derived value of \(\bar{b}\) is consistent with the small asset case scenario we have analyzed throughout, where \(\bar{b}\) is small in the sense that cash investors are unable to purchase the efficient amount of investment good when market liquidity is abundant. The parameter values for the quantitative exercise can be found in Table 1.

We now use the model to assess whether a standing or ad hoc central bank repo facility is better. We vary \(\alpha\) from 0 to 1 and for each \(\alpha\) calculate welfare changes for the optimal central bank repo price, \(p^{CB}_{\alpha}\). Recall that \(\alpha\) represents the fraction of early investors under an emergency repo standing facility scenario that do not have access to central bank repo services. If welfare is maximized at \(\alpha = 0\), then we can conclude that a standing central bank repo facility is best; if welfare is maximized at some interior point, then we can conclude that an ad hoc repo facility is best.

\(^{62}\)For the examples in Section 5 we assume a uniform distribution for \(\sigma\). Since any fixed interval on \([0,1]\) has the same probability of occurring, extreme scarcity of market liquidity would not be viewed as a tail event. We have also experimented several other distributions. The qualitative features described Figure 8 are unchanged by alternative assumptions for the distribution function.
Figure 8: Emergency vs Permanent Facility: Benchmark

Figure 8 illustrates the results from the benchmark parameters. Figure 8(a) shows welfare as a function of $1 - \alpha$, normalizing the value at $\alpha = 1$ to 0. As $1 - \alpha$ increases, welfare increases and is maximized at $\alpha = 0$. This implies that a standing central bank repo facility does better than an ad hoc one when the central bank chooses its repo asset price/borrowing rate optimally.

Figure 8(b) shows a negative relationship between investors’ accumulation of CM subperiod real balances and $1 - \alpha$, the fraction of investors that can access the central bank’s emergency repo facility. When $1 - \alpha$ increases, cash investors will have easier access to the emergency repo facility. Hence, investors have less incentive to accumulate real balances in the CM subperiod since they have a higher probability to access the emergency standing facility for repo finance. An implication is that the CM subperiod asset price, $p^{CM}_\alpha$, increases because bonds have become more liquid.

Figure 8(c) shows that the optimal central bank repo price, $p^{CB}_\alpha$, is a decreasing function of “access” to the emergency repo facility, $1 - \alpha$. As $1 - \alpha$ increases, more cash investors are able to access the emergency repo facility in states where market liquidity is significantly scarce. As a result, if the central bank keeps its repo price fixed, investors will be induced to accumulate less real balances in the CM subperiod. An implication of this outcome is that when market liquidity is not significantly scarce, DM subperiod investment and CM subperiod consumption will be reduced because investors’ real balances are smaller. To mitigate this effect, the central bank decreases its optimal repo asset price, $p^{CB}_\alpha$, when it $1 - \alpha$ increases: a lower central bank repo price reduces FM subperiod liquidity in significantly scarce market liquidity states and investors’ will adjust for this loss by accumulating more real balances in the CM subperiod. Hence, there is a negative relationship between $p^{CB}_\alpha$ and $1 - \alpha$.

\textsuperscript{63}Below, we investigate this relationship if, for some reason, the central bank does not choose the repo price/rate optimally.
Notice that there are two opposing effects on real balance accumulation in the CM subperiod when $1 - \alpha$ is increased. First, holding $p_{CB}^C$ fixed, an increase in access to the central bank’s emergency repo facility reduces investors’ real balance accumulation in the CM subperiod. Second, decreasing the repo asset price, $p_{CB}^C$, while holding access $1 - \alpha$ constant, induces an increase in real balance accumulation in the CM subperiod. Figure 8(b) illustrates that the first effect dominates the second.

Figure 8(d) shows that the probability that the central bank’s emergency repo facility is active is an increasing function of access, $1 - \alpha$. Here, as in the case of investors’ real balance accumulation in the CM subperiod, there are two countervailing forces at work. First, an increase in access, $1 - \alpha$, decreases the central bank repo price $p_{CB}^C$ and, holding all else constant, a decrease in $p_{CB}^C$ increases $\sigma_\alpha$ and, hence, decreases the probability that the emergency facility is used. Second, an increase in access decreases real balance accumulation in the CM subperiod and, holding all else constant, decreasing market liquidity increases the probability that the emergency facility will be used. Figure 8(d) shows that the second effect dominates the first. Interestingly both figures 8(b) and 8(d) have the feature that the “quantity effect,” real balances $z$, dominate the “price effect,” $p_{CB}^C$.

Our numerical example indicates that a standing repo facility delivers better outcomes than an emergency facility, where the latter arises only after significant scarcity in market liquidity is observed in the early FM subperiod. This result depends critically on central bank pursuing the best—welfare maximizing—strategy. The optimal strategy has the central bank posting a very high repo rate, equal to about 12.8%. This rate is “high” in the sense that when liquidity is abundant, the market repo rate is equal to zero. If, for some reason, the central bank does not pursue such a policy, then, depending upon model parameters, an emergency facility may actually outperform a standing repo facility. In terms of our example, suppose that the central bank cannot commit to such a high repo rate and that the maximum repo rate that it will post is 6.3% (for either the standing or emergency facility). For this repo rate, setting $\alpha = 0.47$ maximizes welfare for the emergency repo facility. If the central bank opens an emergency facility after 47% of investors have traded and posts a repo rate of 6.3%, then the emergency repo facility will deliver higher welfare than a standing repo facility that posts a 6.3% repo rate. This implies that, for the parameters in our example and for a central bank that set the repo rate equal to 6.3%, replacing an emergency repo facility with standing facility would reduce welfare. In fact, for central bank repo rates lower than 6.3%, the emergency facility always does
better than a standing facility. Furthermore, if the central bank’s repo rate is set sufficiently low, then it is optimal to set $\alpha = 1$ for the “emergency facility,” i.e., never open the emergency facility. Creating either partial or no access to the central bank facility by using an emergency facility, compared to a standing facility, is beneficial in these examples because the increase in market liquidity—by increasing investors’ CM real balance holdings $z$—more than compensates for the loss in “insurance” associated with the standing repo facility.

Although the distribution of $\sigma$ does not qualitatively affect our results, it does have interesting quantitative implications. In particular, if the distribution has a thicker right tail, then scarce market liquidity is more likely to occur. In this situation, it is optimal for the central bank to intervene more often and more “aggressively” with a lower central bank repo rate. As the right tail thickens, for a fixed $\alpha$ for the emergency facility, the welfare maximizing central bank repo rate falls. And, for a fixed central bank repo rate, a lower $\alpha$ maximizes welfare as the tail thickens. This implies that if the central bank sets the repo rate too low, a permanent repo facility can be optimal if the distribution $\sigma$ has a thick tail, while an emergency facility can be optimal if the tail is thin.

7 Conclusion

Many years ago Bagehot advocated that a lender of last resort should lend freely on good collateral but at a high rate. When liquidity (cash) was very scarce, he believed it important to get cash into the hands of people who need or want to spend it to prevent a downturn in demand and, therefore, economic activity. In this paper we construct an economic environment that takes the spending of these people seriously—meaning there is a demand for a medium of transaction—and find that when the economy is subject to credit or liquidity shocks, the Bagehot prescription, at least qualitatively, is the optimal one. A high lending rate ensures that the private actors, in aggregate, have incentives to accumulate market liquidity so that economic activity is high most of the time and that central bank liquidity provision rarely tapped. Although a lower lending rate may appear to be beneficial for economic outcomes when market liquidity is extremely scarce, a low central bank lending rate, in fact, would lower economic activity on average. If a central bank can commit to the optimal high rate, then it should always be prepared to supply liquidity at the stated rate independent of the state of market liquidity. If, however, such a high rate cannot be committed to, then it may be better for the central bank to actually observe market stress—as expressed by high market rates—before
it supplies any liquidity to the market.

References


Appendix

A Proof to Proposition 1

Suppose that $\pi_H = 1$. We focus on the monetary equilibrium and show that welfare is non-increasing in $p^{CB}$ if the standing repo facility is active. Because $\iota > 0$, two cases can occur.

Case 1: Cash investors do not repo finance all their bonds. Then from (26) and (27), we have

\[
p^{CM} = \beta, \quad \iota = \lambda(z + p^{CB}b^*) = \frac{1}{p^{CB}} - 1.
\]

The first equation implies that the asset is priced at its fundamental value. The second equation implies that this case occurs if and only if $\iota = 1/p^{CB} - 1$. In this situation, a range of $z$ and $b^*$ can occur the equilibrium and they all lead to the same welfare. If $p^{CB}$ increases, then the equilibrium changes to Case 2, where cash investors repo finance all their bonds. If $p^{CB}$ decreases, the standing repo facility is inactive and welfare stays constant.

Case 2: Cash investors repo finance all their bonds. Then the equilibrium $z$ solves

\[
\iota = \sigma_H \lambda (z + p^{CB}b) + (1 - \sigma_H) \left( \frac{1}{p^{CB}} - 1 \right).
\]

This equation implies $z + p^{CB}b$ is decreasing in $p^{CB}$. As a result, a higher $p^{CB}$ leads to lower investment by the cash investor and thus lower welfare.

The above argument shows that an increase in $p^{CB}$ weakly reduces welfare and hence it is optimal to reduce $p^{CB}$ so that the standing facility is not active. This proves the proposition.

B Proof to Proposition 2

To see how investment, consumption and welfare are affected by an active repo facility, we set $p^{CB}$ equal to $p_{FM}^H$ and ask what happens when $p^{CB}$ increases. To understand what happens to welfare, differentiate (30) to get

\[
\frac{dW'}{dp^{CB}} \approx \pi_L \sigma_L \frac{f' \circ Y(\hat{z} + \hat{b}) - c' \circ Y(\hat{z} + \hat{b})}{v' \circ Y(\hat{z} + \hat{b})} \frac{d\hat{z}}{dp^{CB}} + \pi_H \sigma_H \frac{f' \circ Y(\hat{z} + p^{CB}b^*_H) - c' \circ Y(\hat{z} + p^{CB}b^*_H)}{v' \circ Y(\hat{z} + p^{CB}b^*_H)} \frac{d(\hat{z} + p^{CB}b^*_H)}{dp^{CB}}.
\]
where $\approx$ means equal in sign. Using (9), we can rewrite this equation as

$$W'(p^{CB}) \approx \pi_L \sigma_L \lambda(z + \bar{b}) \frac{d\hat{z}}{dp^{CB}} + \pi_H \sigma_H \lambda(z + p^{CB}\hat{b}^*_H) \frac{d(\hat{z} + p^{CB}\hat{b}^*_H)}{dp^{CB}}.$$  

(35)

To evaluate the derivatives on the right side of (35), take the derivatives of (29) and (28) with respect to $p^{CB}$ and rearrange to obtain

$$\frac{d\hat{z}}{dp^{CB}} = \frac{\pi_H}{\sigma_L \pi_L \lambda'(z + \bar{b})(p^{CB})^2} < 0,$$  

(36)

$$\frac{d(\hat{z} + p^{CB}\hat{b}^*_H)}{dp^{CB}} = -\frac{1}{\lambda'(\hat{z} + p^{CB}\hat{b}^*_H)(p^{CB})^2} > 0.$$  

(37)

Since investment in the DM subperiod is monotone in liquidity, we have established part (i) of this proposition.

To establish part (ii), substitute (36) and (37) into (35) to get

$$W'(p^{CB}) \approx \frac{\pi_H}{(p^{CB})^2} \left\{ \frac{\lambda(z + \bar{b})}{\lambda'(z + \bar{b})} - \sigma_H \frac{\lambda(z + p^{CB}\hat{b}^*_H)}{\lambda'(\hat{z} + p^{CB}\hat{b}^*_H)} \right\}.$$

At $p^{CB} = \tilde{p}_H^{FM}$, $\hat{z} = \tilde{z}$ and $\hat{z} + p^{CB}\hat{b}^*_H = \tilde{z}/\sigma_H$. Therefore,

$$W'(\tilde{p}_H^{FM}) \approx \left\{ \frac{\lambda(z + \bar{b})}{\lambda'(z + \bar{b})} - \frac{\lambda(z + \overline{\pi}_H)}{\lambda'(\overline{\pi}_H)} \right\} + (1 - \sigma_H) \frac{\lambda(z + \overline{\pi}_H)}{\lambda'(\overline{\pi}_H)}.$$  

(38)

The first bracketed term in (38) is positive because $\tilde{z} + \bar{b} > \tilde{z}/\sigma_H$ and the second term is negative because $\lambda' < 0$. Therefore, $W'(\tilde{p}_H^{FM}) > 0$ if $\sigma_H$ is sufficiently close to 1.

C  Uniqueness of $\tilde{\sigma}_H$

The following is a sufficient but not necessary condition for $\tilde{\sigma}_H$ to be unique.

**Condition 3** $1 + \lambda(z) + \lambda'(z) z > 0$ for all $z \in [0, z^*]$.

This condition is always satisfied if $\theta = 1$ and $\gamma < 1$. By continuity, it holds for $\theta$ not too small. Indeed, we have checked that in our numerical examples, this condition hold even though $\theta = 0.5$.

We can then prove

**Lemma 4** Assume condition 3 holds. Then in the interval $[z/z^*, 1]$, $\tilde{\sigma}_H$ is unique.
Proof. Recall that $\tilde{\sigma}_H$ is implicitly defined by
\[
\frac{[1 + \lambda (z/\sigma)] z}{[1 + \lambda (z/\sigma)] z + b} = \sigma.
\]
Rearranging, this equation can be written as
\[
(1 - \sigma) \left[ 1 + \lambda \left( \frac{z}{\sigma} \right) \right] \frac{z}{\sigma} = \tilde{b} = \frac{\sigma}{\sigma + b}.
\]
Differentiate the left-hand side with respect to $\sigma$ and obtain
\[
-(1 - \sigma) \lambda' \left( \frac{z}{\sigma} \right) \frac{z}{\sigma^2} \leq -\lambda' \left( \frac{z}{\sigma} \right) \frac{z}{\sigma^2} - \left[ 1 + \lambda \left( \frac{z}{\sigma} \right) \right] \frac{z}{\sigma^2}.
\]
This is equal in sign to
\[
-(1 - \sigma) \lambda' \left( \frac{z}{\sigma} \right) \frac{z}{\sigma^2} \leq -\lambda' \left( \frac{z}{\sigma} \right) \frac{z}{\sigma^2} - \left[ 1 + \lambda \left( \frac{z}{\sigma} \right) \right] \frac{z}{\sigma^2} < 0.
\]
The first inequality holds because $\lambda' < 0$ and $\sigma < 1$. The last inequality holds because by assumption $1 + \lambda (z) + \lambda' (z) z > 0$ for all $z \in [0, z^*]$. As a result, there is at most one solution to (39). ■

D General Model

When $\sigma$ is continuous, the bond pricing and money demand equations, (1) and (2), respectively, for an always inactive central bank repo facility can now be expressed as

\[
p^{CM} = \beta \int_\sigma [\sigma J_1^c (b, z, \sigma) + (1 - \sigma) J_1^a (b, z, \sigma)] dF(\sigma) \tag{40}
\]

and

\[
\frac{\phi}{\phi'} = \beta \int_\sigma [\sigma J_2^c (b, z, \sigma) + (1 - \sigma) J_2^a (b, z, \sigma)] dF(\sigma). \tag{41}
\]

We already have expressions for $J_1^c (b, z, \sigma)$, $J_2^c (b, z, \sigma)$, $J_1^a (b, z, \sigma)$ and $J_2^a (b, z, \sigma)$ when market liquidity is characterized by either scarcity or abundance. When market liquidity is characterized by sufficiency, i.e., when $\sigma = \sigma_M \in (\tilde{\sigma}_L, \tilde{\sigma}_H)$, the cash investor’s FM value function, (7), becomes

\[
J^c (b, z, \sigma_M) = f \circ Y (z + p^{FM}_M b) - P (z + p^{FM}_M b) - b (1 - p^{FM}_M) + W(b, z, 0, 0),
\]

and we have

\[
J_1^c (b, z, \sigma_M) = p^{FM}_M [\lambda (z + p^{FM}_M b) + 1], \tag{42}
\]

\[
J_2^c (b, z, \sigma_M) = \lambda (z + p^{FM}_M b) + 1. \tag{43}
\]
When $\sigma = \sigma_M \in (\hat{\sigma}_L, \hat{\sigma}_H)$, the credit investor’s FM value function, (10), becomes

$$J^n(b, z, \sigma_M) = \left( \frac{1}{p_{FM}^M} - 1 \right) z + W(b, z, 0, 0),$$

and we have

$$J^1_1(b, z, \sigma_M) = 1,$$  \hspace{1cm} (44)

$$J^2_2(b, z, \sigma_M) = \frac{1}{p_{FM}^M}. \hspace{1cm} (45)$$

In equilibrium, when market liquidity is sufficient we have $z + p_{FM}^M \tilde{b} = z/\sigma_M$, which implies that $p_{FM}^M = (1 - \sigma_M)z/(\sigma_M \tilde{b})$. Using (13)-(20) and (42)-(45), the bond pricing and money demand equations (40) and (41) can be simplified to

$$p_{CM} = \beta + \beta \int_{0}^{\hat{\sigma}_L} \sigma \lambda(z + \tilde{b})dF(\sigma) + \beta \int_{\hat{\sigma}_L}^{\hat{\sigma}_H} \sigma \left\{ \frac{(1 - \sigma)z}{\sigma b} \left[ \frac{\lambda(z)}{\sigma} + 1 \right] - 1 \right\} dF(\sigma)$$  \hspace{1cm} (46)

and

$$\iota = \int_{0}^{\hat{\sigma}_L} \sigma \lambda(z + \tilde{b})dF(\sigma) + \int_{\hat{\sigma}_L}^{\hat{\sigma}_H} \left\{ \sigma \lambda(\tilde{z}) \left[ \frac{\sigma b}{(1 - \sigma)z} - 1 \right] \right\} dF(\sigma)$$

\hspace{1cm} + \int_{\hat{\sigma}_H}^{1} \lambda(\tilde{z}) dF(\sigma). \hspace{1cm} (47)$$

Intuitively, cash investors repo finance all of their collateral in the FM subperiod in states $\sigma \in (0, \hat{\sigma}_H)$ and, therefore, get a liquidity benefit from an additional unit of the bond in any of these states. Cash investors get a liquidity benefit from an additional unit of real balances in all states. Credit investors, however, get a liquidity benefit only when $p_{FM}^M < 1$ which occurs in states $\sigma \in (\hat{\sigma}_L, 1)$. Just as in the two-state case, the steady state equilibrium is determined, in part, by two equations in $p_{CM}^M$ and $z$, (46) and (47), which can be solved sequentially. First, the equilibrium $z$ can be solved from (47): because the right side is decreasing in $z$, the equilibrium exists and is unique if $\theta$ is not too small. The equilibrium value of $p_{CM}^M$ is determined by substituting the equilibrium $z$ into (46).

We now assume that the central bank sets its repo price $p_{CB}^M$ so that its repo facility is active in all states $\sigma > \hat{\sigma}$. We start with the situation where $\hat{\sigma}_H$ exists and $\hat{\sigma} > \hat{\sigma}_H$.\textsuperscript{64} The equilibrium one-period government bond price in the CM subperiod, $p_{CM}^M$, is then given by

$$p_{CM}^M = \beta + \beta \int_{0}^{\hat{\sigma}_L} \sigma \lambda(z + \tilde{b})dF(\sigma) + \beta \int_{\hat{\sigma}_L}^{\hat{\sigma}_H} \sigma \left\{ \frac{(1 - \sigma)z}{\sigma b} \left[ \lambda(\tilde{z}) + 1 \right] - 1 \right\} dF(\sigma). \hspace{1cm} (48)$$

\textsuperscript{64}The critical value $\hat{\sigma}_H$ exists if $p_{CB}^M$ is larger than but sufficiently close to $p_{FM}^M = 1/[1 + \lambda(\tilde{z})]$. 49
Notice that, except for the critical value labels, this expression is identical to the bond price equation when the central bank repo facility is always inactive, (46). This should not surprise since an additional unit of the bond confers no additional benefit to investors beyond the fundamental value when market liquidity is scarce. The money demand equation is

\[
\iota = \int_0^{\hat{\sigma}_L} \sigma \lambda (z + \tilde{b}) dF(\sigma) + \int_{\hat{\sigma}_L}^{\hat{\sigma}_H} \left\{ \sigma \lambda \left( \frac{z}{\sigma} \right) + (1 - \sigma) \left[ \frac{\sigma \bar{b}}{(1 - \sigma)z} - 1 \right] \right\} dF(\sigma)
\]

\[
+ \int_{\hat{\sigma}_H}^{\bar{\sigma}} \lambda \left( \frac{z}{\sigma} \right) dF(\sigma) + \frac{1 - p_{CB}}{p_{CB}} [1 - F(\bar{\sigma})].
\]  

(49)

This expression, again except for the critical value labels, is identical to (47) for all states \( \sigma < \hat{\sigma} \). When \( \sigma > \hat{\sigma} \), the repo facility is active and the equilibrium FM subperiod bond, \( p_{\sigma}^{FM} \), is equal to \( p_{CB} \). In this equilibrium cash investors do not access the central bank repo facility when \( \sigma < \hat{\sigma} \) since \( p_{\sigma}^{FM} > p_{CB} \) and when \( \sigma > \hat{\sigma} \), \( b_{\sigma}^c + b_{CB}^c = b^* < \tilde{b} \) for all \( \sigma \). Intuitively, one should think of this case arising when \( p_{CB} \) is not “too big” in the sense that it is “not much” larger than \( \tilde{p}_{1}^{FM} = 1/[1 - \lambda(\tilde{z})] \), the FM subperiod asset price when \( \sigma = 1 \) and the standing facility is always inactive.

When the central bank chooses \( p_{CB} \) “sufficiently large,” then \( \hat{\sigma}_H \) does not exist. In this case, market and central bank liquidity will be sufficient for all \( \sigma > \hat{\sigma} \), meaning that cash investors will repo finance all of their collateral in the FM subperiod in all \( \sigma > \hat{\sigma} \). Since \( \hat{\sigma} > \hat{\sigma}_L \), market plus central bank liquidity will be sufficient for all states \( \sigma > \hat{\sigma}_L \). When \( \hat{\sigma}_H \) does not exist, the critical value \( \bar{\sigma} \) is determined by the equality of the value of cash investors’ repo financing all of their collateral at \( p_{CB} \) per unit collateral, \( \bar{\sigma} p_{CB} \bar{b} \), with the value of market liquidity, \( (1 - \bar{\sigma})z \), i.e.,

\[
\bar{\sigma} = \frac{z}{z + p_{CB} \bar{b}}.
\]

In this situation, the equilibrium government bond price in the CM subperiod, \( p_{CM} \), is

\[
p_{CM} = \beta + \beta \int_0^{\hat{\sigma}_L} \sigma \lambda (z + \tilde{b}) dF(\sigma) + \beta \int_{\hat{\sigma}_L}^{\bar{\sigma}} \left\{ \frac{(1 - \sigma)z}{\sigma \bar{b}} \right\} \left[ \lambda \left( \frac{z}{\sigma} \right) + 1 \right] dF(\sigma)
\]

\[
+ \beta \int_{\bar{\sigma}} \left\{ p_{CB} \left[ \lambda \left( \frac{z}{\sigma} \right) + 1 \right] - 1 \right\} dF(\sigma).
\]  

(50)

The first line of this expression, where \( \sigma < \bar{\sigma} \), is identical to (46) except the cut-off labels. The second line reflects the fact that cash investors continue to repo finance all of their collateral in all states \( \sigma > \bar{\sigma} \) at price \( p_{\sigma}^{FM} = p_{CB} \). (Recall that in the absence of an active repo facility cash investors do not repo finance all of their collateral in states \( \sigma > \hat{\sigma}_H \).)
equation is
\[ \lambda = \int_{0}^{\delta_L} \sigma \lambda(z + \bar{b}) dF(\sigma) + \int_{\delta_L}^{\delta_H} \left\{ \sigma \lambda \left( \frac{z}{\sigma} \right) + (1 - \sigma) \left[ \frac{\sigma \bar{b}}{(1 - \sigma)z} - 1 \right] \right\} dF(\sigma) \]
\[ + \int_{\delta_H}^{1} \left\{ \sigma \lambda \left( \frac{z}{\sigma} \right) + (1 - \sigma) \left[ \frac{1 - P^{CB}}{p^{CB}} \right] \right\} dF(\sigma), \]  
(51)

where this expression is identical to (47) for \( \sigma < \bar{\sigma} \) except the cut-off labels. In states \( \sigma > \bar{\sigma} \), which is described in the second line in the above expression, cash investors repo finance all of their collateral and the equilibrium FM subperiod bond prices are equal to \( p^{CB} \).

Up to this point we have analyzed a standing central bank repo facility. We now discuss and analyze an ad hoc repo facility, where the central bank institutes an emergency facility only if market liquidity is sufficiently scarce (as described in Section 6). With a little abuse of notation, let \( \delta_L \) and \( \delta_H \) be the cutoffs where liquidity becomes just sufficient and scarce in the early FM subperiod. The critical value \( \bar{\sigma} \) describes the set of states where the central bank operates an emergency facility in the late FM subperiod, i.e., when \( \sigma > \bar{\sigma} \). Since \( \bar{\sigma} > \delta_L \), \( \delta_L \) is a relevant statistic for both early and late FM subperiod investors and \( \delta_H \) is a relevant statistic for both the early and late FM subperiod investors if \( \bar{\sigma} > \delta_H \). If \( \delta_H > \bar{\sigma} \), then \( \delta_H \) is only relevant for the early FM subperiod investors as they do not have access to the central bank’s emergency repo facility. First we examine the case where the central bank’s choice of \( p^{CB} \) implies \( \bar{\sigma} > \delta_H \). When \( \sigma > \bar{\sigma} \), the \( 1 - \alpha \) cash investors in the late FM period access the central bank repo facility and repo finance a fraction of their bond holdings, \( \bar{b} \). The bond pricing equation is identical to that associated with a standing repo facility, (48). This should not be surprising since an additional unit of bond does not provide value beyond its fundamental in the FM subperiod in all states \( \sigma > \delta_H \) and we have \( \bar{\sigma} > \delta_H \). The money demand equation is
\[ \lambda = \int_{0}^{\delta_L} \sigma \lambda(z + \bar{b}) dF(\sigma) + \int_{\delta_L}^{\delta_H} \left\{ \sigma \lambda \left( \frac{z}{\sigma} \right) + (1 - \sigma) \left[ \frac{\sigma \bar{b}}{(1 - \sigma)z} - 1 \right] \right\} dF(\sigma) \]
\[ + \int_{\delta_H}^{1} \lambda \left( \frac{z}{\sigma} \right) dF(\sigma) + (1 - \alpha) \left[ \frac{1 - p^{CB}}{p^{CB}} \right] + \alpha \int_{\delta}^{1} \lambda \left( \frac{z}{\sigma} \right) dF(\sigma). \]  
(52)
The only significant difference between this equation and that for the standing repo facility, (49), is in the states where the central bank repo facility is active for the late investors, in states \( \sigma > \bar{\sigma} \). The liquidity premium for real balances with a standing repo facility is equal to \( 1/p^{CB} - 1 \) in all states \( \sigma > \bar{\sigma} \) for all investors. With an ad hoc emergency facility, the liquidity premium is equal to \( 1/p^{CB} - 1 \) in all states \( \sigma > \bar{\sigma} \) but for only those \( 1 - \alpha \) late investors; for the early investors the liquidity premium is given by \( \lambda(z/\sigma) \), which is increasing in \( \sigma > \bar{\sigma} \). Since
\(\lambda(z/\sigma) > 1/p^{CB} - 1\) for all \(\sigma > \bar{\sigma}\), (52) implies that an increase in \(\alpha\), holding \(p^{CB}\) constant, necessarily increases \(z\) and \(\bar{\sigma}\).

We now examine the case where the central bank’s choice of \(p^{CB}\) implies that \(\bar{\sigma} < \hat{\sigma}_H\). In this case, late cash investors repo finance all of their bond \(b\) in the FM market and/or the repo facility for all \(\sigma\). The bond pricing equation is

\[
p^{CM} = \beta + \beta \int_{0}^{\delta_L} \sigma \lambda(z + b) dF(\sigma) + \beta \int_{\delta_L}^{\delta} \sigma \left\{ \left( \frac{1 - \sigma}{\sigma b} \right) \left[ \lambda \left( \frac{z}{\sigma} \right) + 1 \right] - 1 \right\} dF(\sigma) \\
+ (1 - \alpha) \beta \int_{\delta}^{1} \sigma \left\{ \frac{1 - \sigma}{\sigma b} \left[ \lambda \left( \frac{z}{\sigma} \right) + 1 \right] - 1 \right\} dF(\sigma) \\
+ \beta \alpha \int_{\delta}^{\delta_H} \sigma \left\{ \left( \frac{1 - \sigma}{\sigma b} \right) \left[ \lambda \left( \frac{z}{\sigma} \right) + 1 \right] - 1 \right\} dF(\sigma).
\]

This equation is almost identical to the bond pricing equation for the standing repo facility: the first two lines are (almost) identical to (50) except the second line in (53) is multiplied by the fraction, \((1 - \alpha)\), of cash investors that have access to the repo facility in the late FM subperiod. The third line in (53), which is different and reflects the fact that the \(\alpha\) early cash investors do not have access to a central bank repo facility. As a result, these investors repo finance all of their bond in the early FM subperiod in states \(\sigma \in (\bar{\sigma}, \hat{\sigma}_H)\) and repo finance less than \(b\) bond in state \(\sigma > \hat{\sigma}_H\). The money demand equation is

\[
\iota = \int_{0}^{\delta_L} \sigma \lambda(z + b) dF(\sigma) + \int_{\delta_L}^{\delta} \left\{ \sigma \lambda \left( \frac{z}{\sigma} \right) + (1 - \sigma) \left[ \frac{\sigma b}{(1 - \sigma)z} - 1 \right] \right\} dF(\sigma) \\
+ (1 - \alpha) \left\{ \int_{\delta}^{1} \sigma \lambda \left( \frac{z}{\sigma} \right) + (1 - \sigma) \left[ \frac{\sigma b}{(1 - \sigma)z} - 1 \right] \right\} dF(\sigma) \\
+ \alpha \int_{\delta}^{\delta_H} \left\{ \sigma \lambda \left( \frac{z}{\sigma} \right) + (1 - \sigma) \left[ \frac{\sigma b}{(1 - \sigma)z} - 1 \right] \right\} dF(\sigma) + \alpha \int_{\delta}^{1} \lambda \left( \frac{z}{\sigma} \right) dF(\sigma),
\]

which, again, is almost identical to the money demand equation for the standing repo facility (51). Just as above, the last line of (54) does not have a counterpart in (51) and reflects the fact that \(\alpha\) investors do not have access to a standing repo facility in states \(\sigma > \bar{\sigma}\).

### E Large supply of government bonds

We have so far analyzed the case where the supply of government bonds, \(\bar{b}\), is “small.” Here we characterize the threshold of government bond supply that distinguishes the small and large supply cases, and characterize the equilibrium when the central bank repo facility is always inactive and sometime active.

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52
The supply of assets is large when cash investors do not sell all their assets and credit investors do not use all their liquidity to buy assets in the FM. We will show below that a precise definition of the region in which the supply of assets is "large" is given by

\[ \bar{b} \geq z^* - \bar{z}. \]  

(55)

where \( \bar{z} \) solves (59) below.

When cash is abundant, i.e. \( p_{\sigma}^{FM} = 1 \), and cash investors can sell enough assets to purchase the efficient level of DM investment goods, we have \((1 - \sigma) z > \sigma (z^* - z)\). This condition can be rewritten as

\[ \sigma < \frac{z}{z^*}. \]  

(56)

Since cash investors are not constrained by their asset holdings, we must also have

\[ z + \bar{b} \geq z^*. \]  

(57)

In this case the first-order condition (8) holds as an equality. Substituting \( p_{\sigma}^{FM} = 1 \) into (8) we obtain

\[ \lambda (z + b_{\sigma}^c) = 0, \]

which implies that cash investors get enough cash to achieve first best consumption, i.e., \( z + \bar{b} \geq z^* \). Hence, when (56) and (57) hold, use the envelope conditions for the FM subperiod value functions \( J^c \) and \( J^n \) to obtain

\[
\begin{align*}
J^c_1 (b, z, \sigma) &= 1, \\
J^c_2 (b, z, \sigma) &= 0, \\
J^n_1 (b, z, \sigma) &= 1, \\
J^n_2 (b, z, \sigma) &= 0.
\end{align*}
\]

If conditions (56) and (57) are not satisfied, then market liquidity is scarce and the envelope conditions for the value functions \( J^c \) and \( J^n \) are given by (17), (18), (19) and (20). Applying all of these envelope conditions, we are able to write the Euler equations (1) and (2) for the "large" asset case as

\[
\begin{align*}
p^{CM} &= \beta, \\
\iota &= \int_{z/z^*}^{1} \lambda (z/\sigma) dF (\sigma),
\end{align*}
\]

(58)  

(59)

which implies that the government bond is priced at its fundamental value. Equation (59) determines the equilibrium real cash balances \( \bar{z} \). Then given (57), assets are "large" if \( \bar{b} > z^* - \bar{z} \), as we conjectured in equation (55).
E.1 The general model with a standing repo facility

Assume that $b \geq \bar{b}$. Given $p^{CM}$ and $z$, $p^{FM}$ is decreasing in $\sigma$. Hence, there exists a cutoff $\bar{\sigma}$ such that if $\sigma < \bar{\sigma}$, the standing facility is not used and if $\sigma > \bar{\sigma}$, $p^{FM} = p^{CB}$, where $\bar{\sigma}$ solves

$$p^{CB} = \frac{1}{1 + \lambda (z/\bar{\sigma})}.$$ 

Because $p^{CB} \leq 1$, $z/z^{*} < \bar{\sigma}$. Therefore, the Euler equation for money is

$$\ell = \int_{z/z^{*}}^{\bar{\sigma}} \lambda \left( \frac{z}{\sigma} \right) dF(\sigma) + \int_{\bar{\sigma}}^{1} \lambda (z + p^{CB} b^{*}) dF(\sigma),$$

where $b^{*}$ solves $p^{CB} \left[ \lambda (z + p^{CB} b^{*}) + 1 \right] = 1$. The Euler condition for the bond is again given by (58)

$$p^{CM} = \beta.$$ 

E.2 Welfare Implications of a Standing Facility in a Two-State Example

Lastly, we show in a two-state example that a standing repo facility can increase welfare in the large asset case. Suppose $\sigma$ can take value $\sigma_{L}$ with probability $\pi_{L}$ and value $\sigma_{H}$ with probability $\pi_{H} = 1 - \pi_{L}$. Suppose the parametrization is such that $\sigma_{L} < \bar{z}/z^{*}$ and $\sigma_{H} > \bar{z}/z^{*}$, where $\bar{z}$ is the equilibrium real balances that solves

$$\ell = \pi_{H} \lambda (z/\sigma_{H}).$$

The FM price in the high state is $p^{FM}_{H} = 1/ \left[ \lambda (\bar{z}/\sigma_{H}) + 1 \right]$ and that in the low state is $p^{FM}_{L} = 1$. Moreover, $p^{CM} = \beta$ is the equilibrium bond price because cash investors are not asset constrained in all states.

Now we introduce a repo facility with a price $p^{CB}$, which is slightly higher than $p^{FM}_{H}$. Then cash investors are not asset constrained in the high state. Therefore, $\left[ \lambda (z + p^{CB} b^{*}_{H}) + 1 \right] p^{CB} = 1$. Moreover, if $p^{CB} = p^{FM}_{H}$, cash investors are not constrained in the DM. As a result, $\lambda (z_{L}) = 0$ where $z_{L}$ is the real balances held by cash investors in the DM. Therefore,

$$\frac{\partial W \left( p^{CB} \right)}{\partial p^{CB}} \bigg|_{p^{CB} = p^{FM}_{H}} = \frac{1}{\theta} \pi_{H} \sigma_{H} \lambda (z + p^{CB} b^{*}) \frac{\partial \left( z + p^{CB} b^{*} \right)}{\partial p^{CB}} + \frac{1}{\theta} \pi_{L} \lambda \left( z_{L} \right) \frac{\partial z_{L}}{\partial p^{CB}}$$

$$= \frac{1}{\theta} \pi_{H} \sigma_{H} \lambda (z + p^{CB} b^{*}) \frac{\partial \left( z + p^{CB} b^{*} \right)}{\partial p^{CB}}$$

$$= - \frac{1}{\theta} \pi_{H} \sigma_{H} \lambda (z + p^{CB} b^{*}) \frac{1}{\left( p^{CB} \right)^{2}} > 0.$$
Welfare is increasing with $p^{CB}$ if it is close to $\tilde{p}_H^{FM}$. Intuitively, a higher $p^{CB}$ improves investment in the high state but lowers investment in the low state. Because cash investors are making efficient investment in the low state, the improvement in the high state is first order while lower investment in the low state is second order. The net effect on welfare is then positive. Hence we have

**Proposition 5** When the asset supply is “large” in the two-state world, a standing facility can improve welfare.